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# Subgroups of Black-White Point Groups

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The necessity of distinguishing the orientations of a subgroup in a group is pointed out. The numbers of different subgroups and normal subgroups of the black-white point groups are given first by taking into account this distinction, secondly by not distinguishing the orientations, and thirdly by not distinguishing isomorphic subgroups. Groups that can be constructed by successive formation of direct products are assembled into seven families, each of which contains precisely one black-white point group that is not isomorphic to an ordinary point group.

There are 25 non-isomorphic (abstract) black-white point groups that give rise to a total of 122 non-equivalent (crystallographic) black-white point groups (Table 1, column I).

Table 1. *Numbers of isomorphs and subgroups of the abstract black-white point groups*

| | I | II | | III | |
|---|---|---|---|---|---|
| Abstract group | Non-equivalent realizations | Abstract subgroups | Abstract normal subgroups | Total number of subgroups | Total number of normal subgroups |
| $C_1$ | 1 | 1 | 1 | 1 | 1 |
| $C_2$ | 7 | 2 | 2 | 2 | 2 |
| $D_2$ | 12 | 3 | 3 | 5 | 5 |
| $D_{2h}$ | 7 | 4 | 4 | 16 | 16 |
| $D_{2h}\times\theta$ | 1 | 5 | 5 | 67 | 67 |
| $C_4$ | 4 | 3 | 3 | 3 | 3 |
| $C_{4h}$ | 6 | 5 | 5 | 8 | 8 |
| $C_{4h}\times\theta$ | 1 | 7 | 7 | 27 | 27 |
| $D_4$ | 10 | 5 | 5 | 10 | 6 |
| $D_{4h}$ | 9 | 8 | 8 | 35 | 19 |
| $D_{4h}\times\theta$ | 1 | 11 | 11 | 146 | 78 |
| $C_3$ | 1 | 2 | 2 | 2 | 2 |
| $C_6$ | 7 | 4 | 4 | 4 | 4 |
| $C_{6h}$ | 7 | 6 | 6 | 10 | 10 |
| $C_{6h}\times\theta$ | 1 | 7 | 7 | 32 | 32 |
| $D_3$ | 4 | 4 | 3 | 6 | 3 |
| $D_6$ | 16 | 7 | 6 | 16 | 7 |
| $D_{6h}$ | 10 | 10 | 9 | 54 | 21 |
| $D_{6h}\times\theta$ | 1 | 13 | 12 | 236 | 83 |
| $T$ | 1 | 5 | 3 | 10 | 3 |
| $T_h$ | 3 | 8 | 6 | 26 | 6 |
| $T_h\times\theta$ | 1 | 10 | 8 | 88 | 15 |
| $O$ | 4 | 9 | 4 | 30 | 4 |
| $O_h$ | 6 | 16 | 8 | 98 | 9 |
| $O_h\times\theta$ | 1 | 23 | 11 | 420 | 26 |

The 7 black-white point groups that are not isomorphic to an ordinary point group are:

$$D_{2h}\times\theta,\ C_{4h}\times\theta,\ D_{4h}\times\theta,\ C_{6h}\times\theta,\ D_{6h}\times\theta,\ T_h\times\theta,\ O_h\times\theta.$$

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Table 2. *Subgroups that are normal or not, depending on their orientation*

| Group | Subgroup | Number of orientations | Number of normal orientations |
|---|---|---|---|
| $D_4$ | $C_2$ | 5 | 1 |
| $D_{2d}$ | $C_2$ | 3 | 1 |
| $D_4(D_2)$ | $C_2$ | 3 | 1 |
| $D_{2d}(D_2)$ | $C_2$ | 3 | 1 |
| $D_{4h}$ | $C_2$ | 5 | 1 |
| | $S_1$ | 5 | 1 |
| | $C_{2v}$ | 6 | 2 |
| | $C_{2h}$ | 5 | 1 |
| $D_4\times\theta$ | $C_2$ | 5 | 1 |
| | $C_2(C_1)$ | 5 | 1 |
| | $D_2(C_2)$ | 6 | 2 |
| | $C_2\times\theta$ | 5 | 1 |
| $D_{4h}(D_4)$ | $C_2$ | 5 | 1 |
| | $S_1(C_1)$ | 5 | 1 |
| | $C_{2v}(C_2)$ | 6 | 2 |
| | $C_{2h}(C_2)$ | 5 | 1 |
| $D_{4h}(D_{2h})$ | $C_2$ | 3 | 1 |
| | $S_1$ | 3 | 1 |
| | $C_{2v}$ | 3 | 1 |
| | $C_{2h}$ | 3 | 1 |
| $D_{2d}\times\theta$ | $C_2$ | 3 | 1 |
| | $C_2(C_1)$ | 3 | 1 |
| | $D_2(C_2)$ | 3 | 1 |
| | $C_2\times\theta$ | 3 | 1 |
| $D_{4h}(D_{2d})$ | $C_2$ | 3 | 1 |
| | $S_1(C_1)$ | 3 | 1 |
| | $C_{2v}(C_2)$ | 3 | 1 |
| | $C_{2h}(C_2)$ | 3 | 1 |
| $D_{4h}\times\theta$ | $C_2$ | 5 | 1 |
| | $S_1$ | 5 | 1 |
| | $C_2(C_1)$ | 5 | 1 |
| | $S_1(C_1)$ | 5 | 1 |
| | $C_{2v}$ | 6 | 2 |
| | $D_2(C_2)$ | 6 | 2 |
| | $C_{2v}(C_2)$ | 6 | 2 |
| | $C_{2h}$ | 5 | 1 |
| | $C_2\times\theta$ | 5 | 1 |
| | $C_{2h}(C_2)$ | 5 | 1 |
| | $C_{2h}(S_2)$ | 5 | 1 |
| | $S_1\times\theta$ | 5 | 1 |
| | $C_{2h}(S_1)$ | 5 | 1 |
| | $C_{2h}\times\theta$ | 5 | 1 |

Table 2 (*cont.*)

| Group | Subgroup | Number of orientations | Number of normal orientations |
|---|---|---|---|
| $D_6$ | $C_2$ | 7 | 1 |
| $D_{3h}$ | $S_1$ | 4 | 1 |
| $D_6(D_3)$ | $C_2(C_1)$ | 4 | 1 |
| $D_{3h}(D_3)$ | $S_1(C_1)$ | 4 | 1 |
| $D_{6h}$ | $C_2$ | 7 | 1 |
| | $S_1$ | 7 | 1 |
| | $C_{2h}$ | 7 | 1 |
| $D_6\times\theta$ | $C_2$ | 7 | 1 |
| | $C_2(C_1)$ | 7 | 1 |
| | $C_2\times\theta$ | 7 | 1 |
| $D_{6h}(D_6)$ | $C_2$ | 7 | 1 |
| | $S_1(C_1)$ | 7 | 1 |
| | $C_{2h}(C_2)$ | 7 | 1 |
| $D_{6h}(D_{3d})$ | $C_2(C_1)$ | 4 | 1 |
| | $S_1(C_1)$ | 4 | 1 |
| | $C_{2h}(S_2)$ | 4 | 1 |
| $D_{3h}\times\theta$ | $S_1$ | 4 | 1 |
| | $S_1(C_1)$ | 4 | 1 |
| | $S_1\times\theta$ | 4 | 1 |
| $D_{6h}(D_{3h})$ | $S_1$ | 4 | 1 |
| | $C_2(C_1)$ | 4 | 1 |
| | $C_{2h}(S_1)$ | 4 | 1 |

Table 3(*a*). *Subgroups of the black-white point groups*

S means number of non-equivalent subgroups
N means number of non-equivalent normal subgroups

| N | S | POINT GROUP | | SUB-GROUP |
|---|---|---|---|---|
| 1 | 1 | 1 | $C_1$ | 1 |
| 2 | 2 | 2 | $C_2$ | 2 |
| 2 | 2 | m | $S_1$ | 3 |
| 2 | 2 | 2' | $C_2(C_1)$ | 4 |
| 2 | 2 | m' | $S_1(C_1)$ | 5 |
| 2 | 2 | $\bar{1}$ | $S_2$ | 6 |
| 2 | 2 | 11' | $C_1\times\theta$ | 7 |
| 2 | 2 | $\bar{1}$' | $S_2(C_1)$ | 8 |
| 3 | 3 | 222 | $D_2$ | 9 |
| 4 | 4 | mm2 | $C_{2v}$ | 10 |
| 4 | 4 | 2'2'2 | $D_2(C_2)$ | 11 |
| 4 | 4 | m'm'2 | $C_{2v}(C_2)$ | 12 |
| 5 | 5 | 2/m | $C_{2h}$ | 13 |
| 5 | 5 | 21' | $C_2\times\theta$ | 14 |
| 5 | 5 | 2/m' | $C_{2h}(C_2)$ | 15 |
| 5 | 5 | 2'/m' | $C_{2h}(S_2)$ | 16 |
| 5 | 5 | m1' | $S_1\times\theta$ | 17 |
| 5 | 5 | 2'/m | $C_{2h}(S_1)$ | 18 |
| 5 | 5 | m'm2' | $C_{2v}(S_1)$ | 19 |
| 5 | 5 | $\bar{1}$1' | $S_2\times\theta$ | 20 |
| 8 | 8 | mmm | $D_{2h}$ | 21 |
| 8 | 8 | 2221' | $D_2\times\theta$ | 22 |
| 8 | 8 | m'm'm' | $D_{2h}(D_2)$ | 23 |
| 12 | 12 | m'm'm | $D_{2h}(C_{2v})$ | 24 |
| 12 | 12 | mm 21' | $C_{2v}\times\theta$ | 25 |
| 12 | 12 | mm m' | $D_{2h}(C_{2v})$ | 26 |
| 16 | 16 | 2/m 1' | $C_{2h}\times\theta$ | 27 |
| 28 | 28 | mmm1' | $D_{2h}\times\theta$ | 28 |
| 3 | 3 | 4 | $C_4$ | 29 |
| 3 | 3 | $\bar{4}$ | $S_4$ | 30 |
| 3 | 3 | 4' | $C_4(C_2)$ | 31 |
| 3 | 3 | $\bar{4}$' | $S_4(C_2)$ | 32 |
| 8 | 8 | 4/m | $C_{4h}$ | 33 |
| 8 | 8 | 41' | $C_4\times\theta$ | 34 |
| 8 | 8 | 4/m' | $C_{4h}(C_4)$ | 35 |
| 8 | 8 | 4'/m | $C_{4h}(C_{2h})$ | 36 |
| 8 | 8 | $\bar{4}$1' | $S_4\times\theta$ | 37 |
| 8 | 8 | 4'/m' | $C_{4h}(S_4)$ | 38 |
| 27 | 27 | 4/m 1' | $C_{4h}\times\theta$ | 39 |
| 5 | 5 | 422 | $D_4$ | 40 |
| 6 | 7 | $\bar{4}$2 m | $D_{2d}$ | 41 |
| 6 | 7 | 4'2'2 | $D_4(D_2)$ | 42 |
| 6 | 7 | $\bar{4}$'2 m' | $D_{2d}(D_2)$ | 43 |
| 5 | 6 | 4mm | $C_{4v}$ | 44 |
| 5 | 6 | 42'2' | $D_4(C_4)$ | 45 |
| 5 | 6 | 4 m'm' | $C_{4v}(C_4)$ | 46 |
| 6 | 8 | $\bar{4}$2'm' | $D_{2d}(S_4)$ | 47 |
| 6 | 8 | 4'm'm' | $C_{4v}(C_{2v})$ | 48 |
| 6 | 8 | $\bar{4}$'2'm | $D_{2d}(C_{2v})$ | 49 |
| 15 | 15 | 4/mmm | $D_{4h}$ | 50 |
| 15 | 15 | 4221' | $D_4\times\theta$ | 51 |
| 15 | 15 | 4/m'm'm' | $D_{4h}(D_4)$ | 52 |
| 19 | 23 | 4/mm m' | $D_{4h}(D_{2h})$ | 53 |
| 19 | 23 | $\bar{4}$2m1' | $D_{2d}\times\theta$ | 54 |
| 19 | 23 | 4/m'm'm | $D_{4h}(D_{2d})$ | 55 |
| 15 | 19 | 4/m m'm' | $D_{4h}(C_{4v})$ | 56 |
| 15 | 19 | 4mm1' | $C_{4v}\times\theta$ | 57 |
| 15 | 19 | 4/m'mm | $D_{4h}(C_{4v})$ | 58 |
| 58 | 59 | 4/mmm1' | $D_{4h}\times\theta$ | 59 |

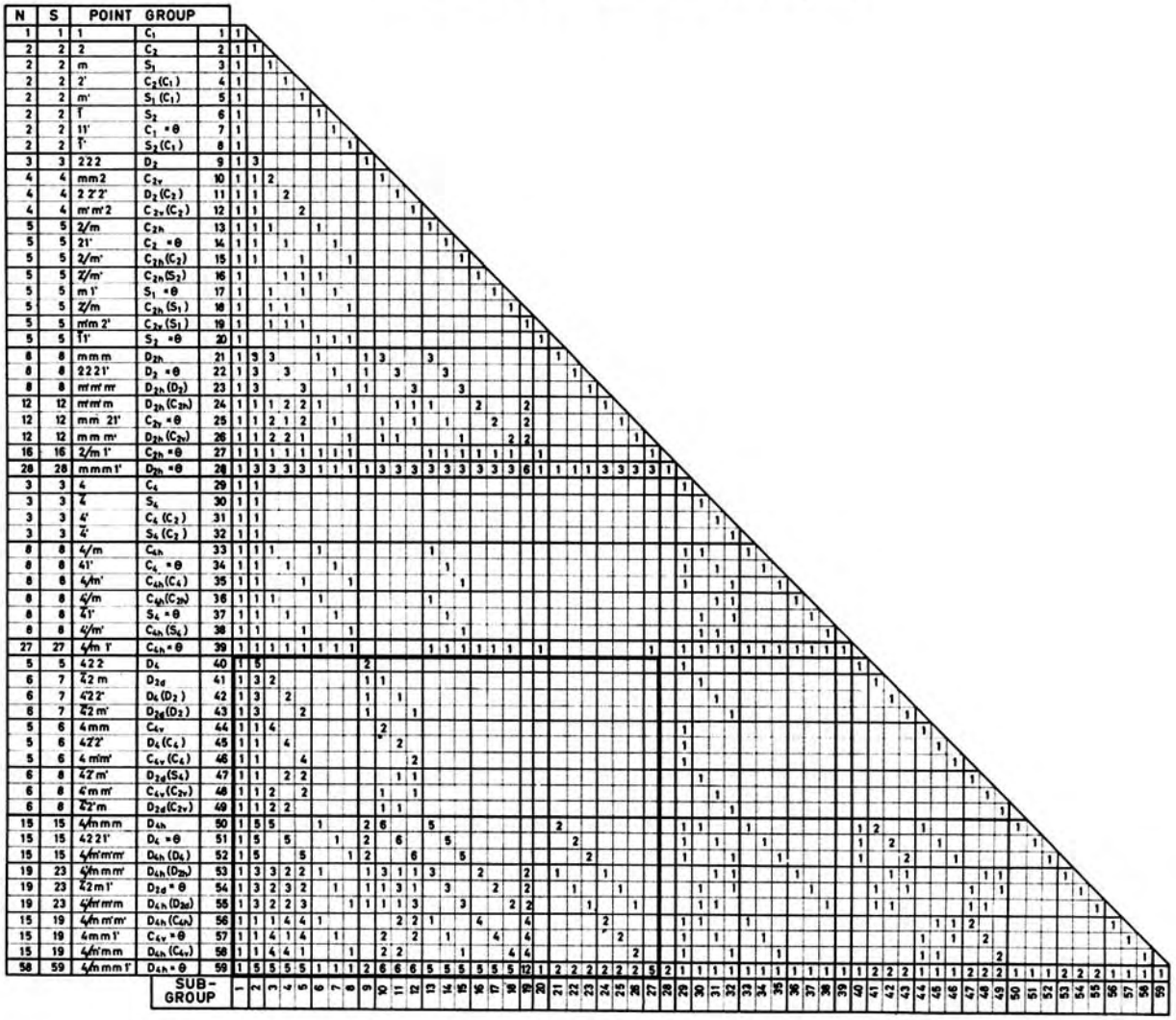


Table 2 (*cont.*)

| Group | Subgroup | Number of orientations | Number of normal orientations | Group | Subgroup | Number of orientations | Number of normal orientations |
|---|---|---|---|---|---|---|---|
| $D_{6h}\times\theta$ | $C_2$ | 7 | 1 | $O_h$ | $D_2$ | 4 | 1 |
| | $S_1$ | 7 | 1 | | $D_{2h}$ | 4 | 1 |
| | $C_2(C_1)$ | 7 | 1 | $O\times\theta$ | $D_2$ | 4 | 1 |
| | $S_1(C_1)$ | 7 | 1 | | $D_2\times\theta$ | 4 | 1 |
| | $C_{2h}$ | 7 | 1 | $O_h(O)$ | $D_2$ | 4 | 1 |
| | $C_2\times\theta$ | 7 | 1 | | $D_{2h}(D_2)$ | 4 | 1 |
| | $C_{2h}(C_2)$ | 7 | 1 | $O_h\times\theta$ | $D_2$ | 4 | 1 |
| | $C_{2h}(S_2)$ | 7 | 1 | | $D_{2h}$ | 4 | 1 |
| | $S_1\times\theta$ | 7 | 1 | | $D_2\times\theta$ | 4 | 1 |
| | $C_{2h}(S_1)$ | 7 | 1 | | $D_{2h}(D_2)$ | 4 | 1 |
| | $C_{2h}\times\theta$ | 7 | 1 | | $D_{2h}\times\theta$ | 4 | 1 |
| $O$ | $D_2$ | 4 | 1 | | | | |

Table 3(*b*). *Subgroups of the black-white point groups*

S means number of non-equivalent subgroups
N means number of non-equivalent normal subgroups

| N | S | POINT GROUP | | |
|---|---|---|---|---|
| 2 | 2 | 3 | $C_3$ | 60 |
| 4 | 4 | $\bar{6}$ | $C_{3h}$ | 61 |
| 4 | 4 | $\bar{6}'$ | $C_{3h}(C_3)$ | 62 |
| 4 | 4 | $\bar{6}'$ | $C_{3h}(C_3)$ | 63 |
| 4 | 4 | $\bar{6}'$ | $C_{3h}(C_3)$ | 64 |
| 4 | 4 | $\bar{3}$ | $S_6$ | 65 |
| 4 | 4 | $31'$ | $C_3\times\theta$ | 66 |
| 4 | 4 | $\bar{3}'$ | $S_6(C_3)$ | 67 |
| 10 | 10 | 6/m | $C_{6h}$ | 68 |
| 10 | 10 | $61'$ | $C_6\times\theta$ | 69 |
| 10 | 10 | 6/m′ | $C_{6h}(C_6)$ | 70 |
| 10 | 10 | 6/m′ | $C_{6h}(S_6)$ | 71 |
| 10 | 10 | $\bar{6}1'$ | $C_{3h}\times\theta$ | 72 |
| 10 | 10 | 6/m | $C_{6h}(C_{3h})$ | 73 |
| 10 | 10 | $31'$ | $S_6\times\theta$ | 74 |
| 32 | 32 | 6/m 1′ | $C_{6h}\times\theta$ | 75 |
| 3 | 4 | 32 | $D_3$ | 76 |
| 3 | 4 | 3 m | $C_{3v}$ | 77 |
| 3 | 4 | 32′ | $D_3(C_3)$ | 78 |
| 3 | 4 | 3 m′ | $C_{3v}(C_3)$ | 79 |
| 6 | 7 | 622 | $D_6$ | 80 |
| 7 | 10 | $\bar{3}$ m | $D_{3d}$ | 81 |
| 7 | 10 | $321'$ | $D_3\times\theta$ | 82 |
| 7 | 10 | $\bar{3}'$ m′ | $D_{3d}(D_3)$ | 83 |
| 7 | 10 | $\bar{3}$ m′ | $D_{3d}(S_6)$ | 84 |
| 7 | 10 | 3 m1′ | $C_{3v}\times\theta$ | 85 |
| 7 | 10 | $\bar{3}'$ m | $D_{3d}(C_{3v})$ | 86 |
| 7 | 9 | $\bar{6}$ m2 | $D_{3h}$ | 87 |
| 7 | 9 | 6′22′ | $D_6(D_3)$ | 88 |
| 7 | 9 | $\bar{6}'$ m′2 | $D_{3h}(D_3)$ | 89 |
| 6 | 8 | 6 mm | $C_{6v}$ | 90 |
| 6 | 8 | 62′2′ | $D_6(C_6)$ | 91 |
| 6 | 8 | 6 m′m′ | $C_{6v}(C_6)$ | 92 |
| 7 | 10 | $\bar{6}$ m′2′ | $D_{3h}(C_{3h})$ | 93 |
| 7 | 10 | 6′mm′ | $C_{6v}(C_{3v})$ | 94 |
| 7 | 10 | $\bar{6}'$ m′2′ | $D_{3h}(C_{3v})$ | 95 |
| 17 | 20 | 6/mmm | $D_{6h}$ | 96 |
| 17 | 20 | $6221'$ | $D_6\times\theta$ | 97 |
| 17 | 20 | 6/m′m′m′ | $D_{6h}(D_6)$ | 98 |
| 21 | 28 | 6/m′m′m′ | $D_{6h}(D_{3d})$ | 99 |
| 21 | 28 | $\bar{6}$ m 21′ | $D_{3h}\times\theta$ | 100 |
| 21 | 28 | 6/mmm′ | $D_{6h}(D_{3h})$ | 101 |
| 17 | 24 | 6/mm′m′ | $D_{6h}(C_{6v})$ | 102 |
| 17 | 24 | 6 mm1′ | $C_{6v}\times\theta$ | 103 |
| 17 | 24 | 6/m′mm | $D_{6h}(C_{6v})$ | 104 |
| 21 | 32 | $\bar{3}$ m1′ | $D_{3d}\times\theta$ | 105 |
| 63 | 75 | 6/mmm1′ | $D_{6h}\times\theta$ | 106 |
| 3 | 5 | 23 | $T$ | 107 |
| 6 | 12 | m3 | $T_h$ | 108 |
| 6 | 12 | $231'$ | $T\times\theta$ | 109 |
| 6 | 12 | m′3 | $T_h(T)$ | 110 |
| 15 | 37 | m31′ | $T_h\times\theta$ | 111 |
| 4 | 9 | 432 | $O$ | 112 |
| 4 | 11 | $\bar{4}3$m | $T_d$ | 113 |
| 4 | 11 | $43'2'$ | $O(T)$ | 114 |
| 4 | 11 | $\bar{4}'3$m′ | $T_d(T)$ | 115 |
| 9 | 25 | m3m | $O_h$ | 116 |
| 9 | 25 | $4321'$ | $O\times\theta$ | 117 |
| 9 | 25 | m′3m′ | $O_h(O)$ | 118 |
| 9 | 33 | m3m′ | $O_h(T_h)$ | 119 |
| 9 | 33 | $\bar{4}3$ m1′ | $T_d\times\theta$ | 120 |
| 9 | 33 | m′3m | $O_h(T_d)$ | 121 |
| 26 | 91 | m3m1′ | $O_h\times\theta$ | 122 |

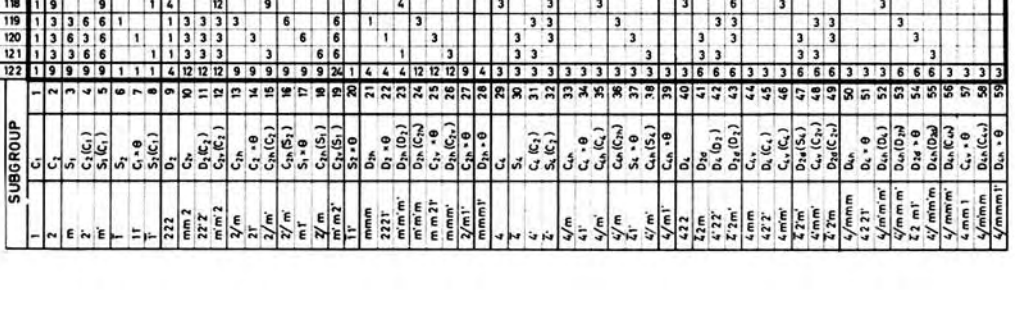
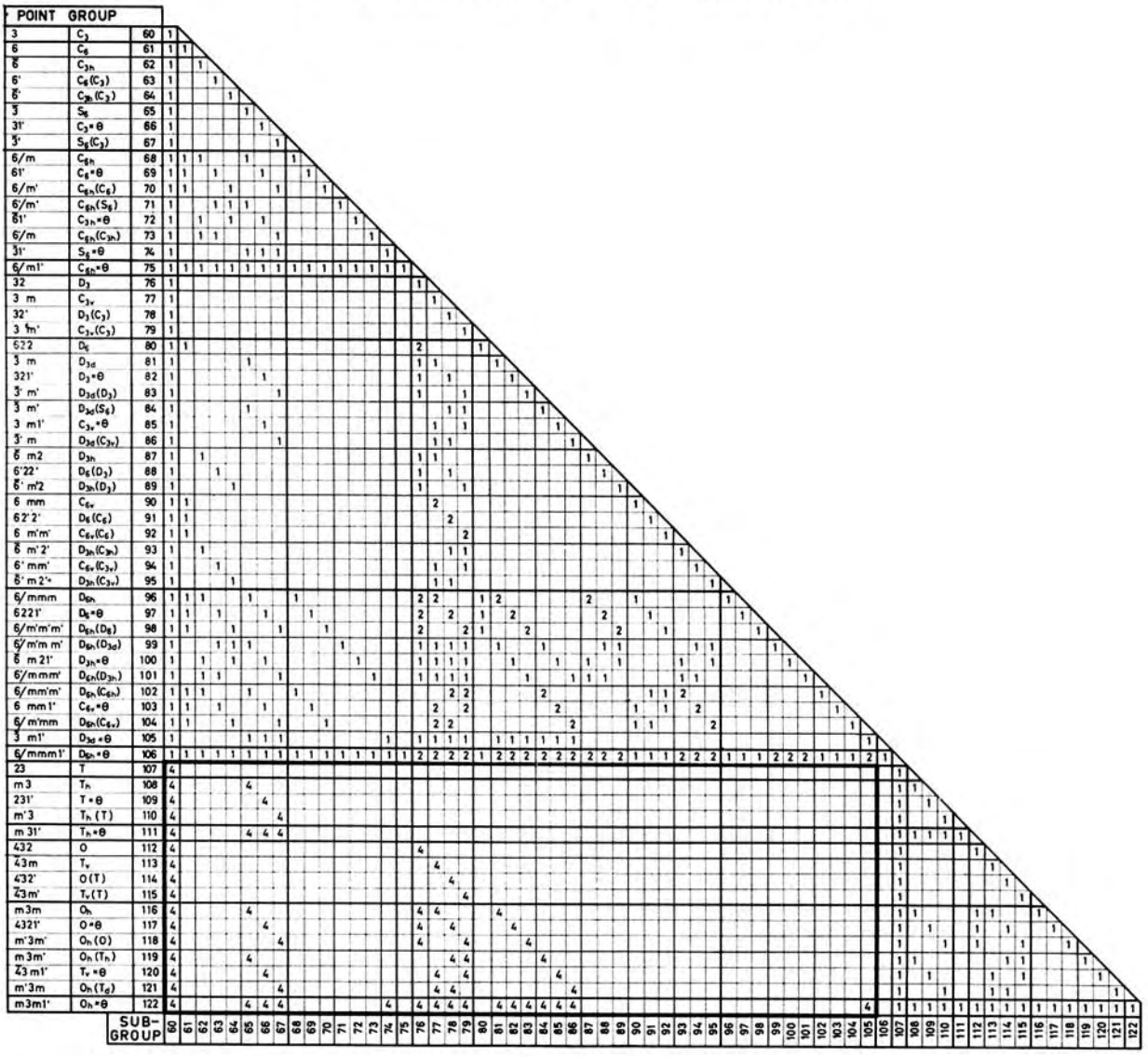


Table 3(*c*). *Subgroups of the black-white point groups*



$\theta$ is the group consisting of the identity and of the colour reversal operators.

The usual notation $G(H)$ for black-white point group means that it has a subgroup $H$ of index two, containing all elements that are not multiplied by the colour reversal operator (see *e.g.* Hamermesh, 1962).

When considering the subgroups of the black-white point groups, it is advisable to distinguish not only the different crystallographic black-white point groups but also the different orientations that may occur with respect to a given group. There are several reasons for doing so.

Isomorphic groups must have the same number of subgroups. Thus the abstract group $D_{2h}$ has 4 non-isomorphic subgroups: $C_1$, $C_2$, $D_2$, and $D_{2h}$ all of which are normal. If we distinguish all non-equivalent black-white point groups but not the various orientations, there are 7 groups isomorphic to $D_{2h}$: $D_{2h}$, $D_2 \times \theta$ and $D_{2h}(D_2)$ each have 8 subgroups, whereas $D_{2h}(C_{2h})$, $C_{2v} \times \theta$, and $D_{2h}(C_{2v})$ have each 12 subgroups; $C_{2h} \times \theta$ even has 16 of them. If now we distinguish between the different orientations, each of the 7 groups again has the same number of subgroups, *viz.* 16 that are all normal. In Table 1 we indicate the number of non-isomorphic subgroups and non-isomorphic normal subgroups (in column II), and the total number of different subgroups and different normal subgroups (in column III) for each of the 25 abstract black-white point groups. The number of subgroups without distinction of orientation within the group may be found in Tables 3(*a*) and 3(*b*).

A more stringent reason appears when we wish to single out the normal subgroups; this can generally

Table 4(*a*). *Normal subgroups within the heavy frame of Table* 3(*a*)

| POINT GROUP | | | SUBGROUP | 1 | $C_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | 2 | $C_2$ | 1 | 1 | | | | | | | 2 | | | | | | | | | | | | | | | | | | |
| $\bar{4}2m$ | $D_{2d}$ | 41 | | $S_1$ | $C_2(C_1)$ | $S_1(C_1)$ | $S_2$ | $C_1\times\theta$ | $S_2(C_1)$ | $D_2$ | $C_{2v}$ | $D_2(C_2)$ | $C_{2v}(C_2)$ | $C_{2h}$ | $C_2\times\theta$ | $C_{2h}(C_2)$ | $C_{2h}(S_2)$ | $S_1\times\theta$ | $C_{2h}(S_1)$ | $C_{2v}(S_1)$ | $S_2\times\theta$ | $D_{2h}$ | $D_2\times\theta$ | $D_{2h}(D_2)$ | $D_{2h}(C_{2h})$ | $C_{2v}\times\theta$ | $D_{2h}(C_{2v})$ | $C_{2h}\times\theta$ |
| | | | | m | 2' | m' | $\bar{1}$ | 11' | $\bar{1}'$ | 222 | mm2 | 22'2' | m'm'2 | 2/m | 21' | 2/m' | 2'/m' | m1' | 2'/m | m'm2' | $\bar{1}1'$ | mmm | 2221' | m'm'm' | m'm'm | m m 21' | m m m' | 2/m1' |
| 422 | $D_4$ | 40 | | 1 | 1 | | | | | | | | | 2 | | | | | | | | | | | | | | | | | | |
| $\bar{4}2m$ | $D_{2d}$ | 41 | | 1 | 1 | | | | | | | | | 1 | 1 | | | | | | | | | | | | | | | | | |
| 4'22' | $D_4(D_2)$ | 42 | | 1 | 1 | | | | | | | | | 1 | | 1 | | | | | | | | | | | | | | | | |
| $\bar{4}2m'$ | $D_{2d}(D_2)$ | 43 | | 1 | 1 | | | | | | | | | 1 | | | 1 | | | | | | | | | | | | | | | |
| 4mm | $C_{4v}$ | 44 | | 1 | 1 | | | | | | | | | | 2 | | | | | | | | | | | | | | | | | |
| 42'2' | $D_4(C_4)$ | 45 | | 1 | 1 | | | | | | | | | | | 2 | | | | | | | | | | | | | | | | |
| 4m'm' | $C_{4v}(C_4)$ | 46 | | 1 | 1 | | | | | | | | | | | | 2 | | | | | | | | | | | | | | | |
| $\bar{4}2'm'$ | $D_{2d}(S_4)$ | 47 | | 1 | 1 | | | | | | | | | | | 1 | 1 | | | | | | | | | | | | | | | |
| 4'mm' | $C_{4v}(C_{2v})$ | 48 | | 1 | 1 | | | | | | | | | | 1 | | 1 | | | | | | | | | | | | | | | |
| $\bar{4}2'm$ | $D_{2d}(C_{2v})$ | 49 | | 1 | 1 | | | | | | | | | | 1 | 1 | | | | | | | | | | | | | | | | |
| 4/mmm | $D_{4h}$ | 50 | | 1 | 1 | 1 | | | 1 | | | 2 | 2 | | | | 1 | | | | | | | | 2 | | | | | | | |
| 4221' | $D_4\times\theta$ | 51 | | 1 | 1 | | 1 | | | 1 | | | 2 | | 2 | | | 1 | | | | | | | | 2 | | | | | | |
| 4/m'm'm' | $D_{4h}(D_4)$ | 52 | | 1 | 1 | | | 1 | | | 1 | | 2 | | | 2 | | | 1 | | | | | | | | 2 | | | | | |
| 4'/mm m' | $D_{4h}(D_{2h})$ | 53 | | 1 | 1 | 1 | | | 1 | | | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | 1 | | | 1 | | | | |
| $\bar{4}2m1'$ | $D_{2d}\times\theta$ | 54 | | 1 | 1 | | 1 | | | 1 | | | 1 | 1 | 1 | 1 | | 1 | | | | | | | | 1 | | | 1 | | | |
| 4'/m'm'm | $D_{4h}(D_{2d})$ | 55 | | 1 | 1 | | | 1 | | | 1 | | 1 | 1 | 1 | 1 | | | 1 | | | | | | | | 1 | | | 1 | | |
| 4/mm'm' | $D_{4h}(C_{4h})$ | 56 | | 1 | 1 | 1 | | | 1 | | | | | | | 2 | 2 | 1 | | | | | | | | | | 2 | | | | |
| 4mm1' | $C_{4v}\times\theta$ | 57 | | 1 | 1 | | 1 | | | 1 | | | 2 | | 2 | | 2 | | 1 | | | | | | | | | | 2 | | | |
| 4/m'mm | $D_{4h}(C_{4v})$ | 58 | | 1 | 1 | | | 1 | | | 1 | | 1 | | 2 | 2 | | | | 1 | | | | | | | | | | 2 | | |
| 4/mmm1' | $D_{4h}\times\theta$ | 59 | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 1 |

not be done if the orientation of the subgroup is not specified. In $D_{4h}$ for instance the subgroup $C_{2v}$ occurs in 6 different orientations; two of them give rise to a normal subgroup, four of them do not. Table 2 gives a complete list of groups with subgroups that are normal or not according to their orientation.

In Tables 3(*a*), 3(*b*), 3(*c*) we indicate the number of times each subgroup occurs in one of the 122 black-white point groups. All subgroups outside the heavy frame are normal. The normal subgroups occurring within the heavy frame are indicated in Tables 4(*a*) and 4(*b*). So far as Table 3(*c*) is concerned, none of the subgroups within the heavy frame is normal.

One word about the order we have adopted for the enumeration of groups. First of all, isomorphic groups are placed together. Furthermore, groups that can be constructed by successive formation of direct products with the abstract group of order two are assembled together. Besides $C_1$ we thus have the 7 families:

$C_2$, $D_2=C_2^z\times C_2^y$, $D_{2h}$, $D_{2h}\times\theta$
$C_4$, $C_{4h}$, $C_{4h}\times\theta$
$D_4$, $D_{4h}$, $D_{4h}\times\theta$
$C_3$, $C_6^z=C_3^z\times C_2^z$, $C_{6h}$, $C_{6h}\times\theta$
$D_3$, $D_6^z=D_3^z\times C_2^z$, $D_{6h}$, $D_{6h}\times\theta$
$T$, $T_h$, $T_h\times\theta$
$O$, $O_h$, $O_h\times\theta$.

Each family contains precisely one black-white point group that is not isomorphic to an ordinary point group.

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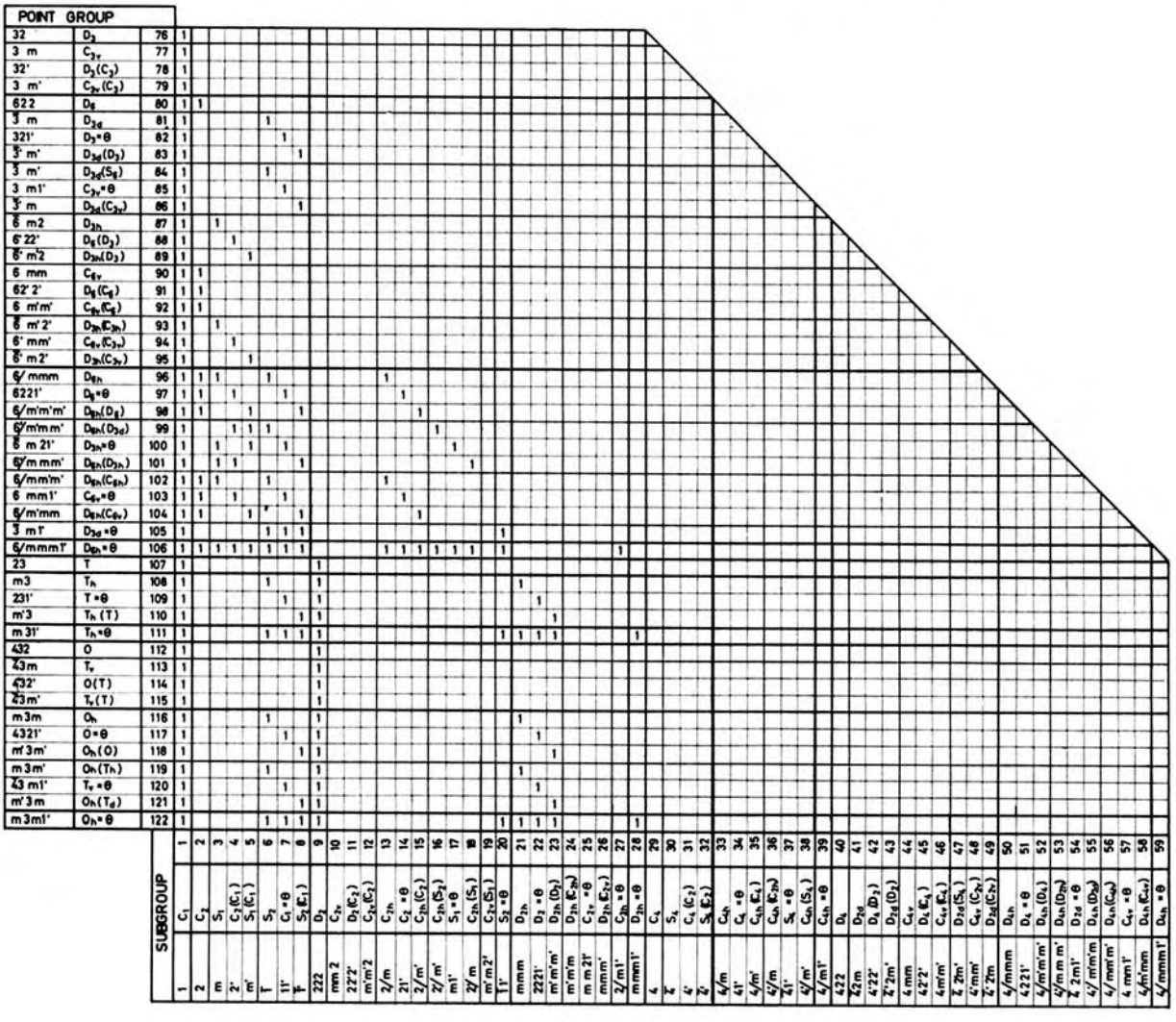
Table 4(*b*). *Normal subgroups within the heavy frame of Table* 3(*b*)



## References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