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BRAVAIS CLASSES OF TWO-DIMENSIONAL RELATIVISTIC LATTICES

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Synopsis

Repetition of motifs in space and time gives rise to regular patterns with symmetries described by relativistic crystallographic groups. This leads to natural generalizations of concepts familiar in Euclidean crystallography. Here only the two-dimensional relativistic case is considered. Conditions are derived for a Lorentz transformation to be crystallographic, *i.e.* to leave invariant a lattice in the two-dimensional Minkowskian space. The introduction of crystallographic transformations that change the sign of the indefinite metric tensor appears to be a necessary step in relativistic crystallography. The corresponding concepts of the theory of binary quadratic forms and of real quadratic fields are briefly discussed.

A classification of all possible relativistic two-dimensional lattices is given and the corresponding Bravais classes are derived (at least in principle, as there are an infinite number of them). Isotropic lattices (*i.e.* with lattice points on the light cone) and incommensurable lattices (*i.e.* with incommensurable metric tensor) have as holohedry a point group of finite order. The other ones, which are described essentially by metric tensors $g(B) = (a, b, c)$ with relatively prime rational integers a, b, c and discriminant $d = b^2 - 4ac$ not a square, always have a holohedry of infinite order. A number of lattice representatives of Bravais classes ordered according to the kinematical interpretation of the Lorentz transformation is given in the appendix.

1. *Introduction*. Periodic repetition of an arbitrary (finite) motif in space gives rise to patterns having crystallographic space groups as symmetry groups[1,2]. Homogeneity of empty space, or in other words, covariancy of physical laws with respect to the Euclidean translation group, is a necessary condition for the existence of crystals. This property, which makes possible the existence of the same motif of matter (*e.g.* molecule) at different places, is of course not sufficient for ensuring the existence of crystals, but makes this existence not at all surprising.

Actually physical laws are covariant with respect to the inhomogeneous Lorentz group, which includes the Euclidean translations as a subgroup. In the same way as above, this fundamental property makes very plausible the existence in nature of periodic repetition of some (finite) motif in space and time. We call such four-dimensional patterns *space-time crystals*, which then have the symmetry of relativistic space-time groups[3]).

Periodic motions on the one hand, and static (three-dimensional) crystals on the other, are degenerate cases of space-time crystals, as they give rise to infinite space-time unit cells. To get a finite unit cell, one may consider the combination of the two previous examples in the form of some crystal vibrating in a given mode, which then has the symmetry of a space-time group. From a general point of view, however, this example may represent too simple a case.

Further analysis of the problem of finding out the possible physical manifestations of space-time crystals shows that one has to look into kinematic symmetries in distributions of matter (like *e.g.* those of conduction electrons in metals). It is then clear that if the typical velocities occurring in such distributions with respect to the centre of mass are small compared to the velocity of light, one has to consider the non-relativistic limit, *i.e.* what we call Galilean space-time groups. These groups have very peculiar features, and will form the subject of a subsequent paper.

Our lack of a synthetic intuition of physical phenomena in space and time together, and the incredible richness of relativistic crystallography (as compared with the Euclidean one) make it advisable not to start thinking about possible space-time crystals before having some basic knowledge of space-time groups[3,4]). In our opinion this is also true if one is interested in another fundamental question, namely that of the existence of a fundamental length in nature. If such a length does exist and space-time is homogeneous no more, then one expects that some type of crystallographic description of matter in space-time is not only possible but even necessary. This is in fact what can be observed already in the current literature where a lattice is associated to the fundamental length[5-9]). However, we should like to underline the fact that matter can very well be arranged according to the symmetry of space-time groups even if there is no fundamental length.

We begin now to have a first view of relativistic crystallography and this paper is the first one devoted to the two-dimensional case (one time-dimension, one space-dimension). In two dimensions everthing can be done explicitly. This is no more the case in higher dimensions, in which (for the moment) only some families of groups can be derived.

The interesting point is that even in two dimensions and because of the richness of the indefinite metric case, it is much more easy to grasp general laws than in the positive definite case. More than that, a two-dimensional crystallography appears which may be formulated completely in terms of arithmetic functions independently of the character of the metric. We consider this as a main result. Even the case where the determinant of the metric tensor vanishes, which arises in the limiting case of Galilean space-time, is correctly described in such an approach. In this paper, however, we consider only the relativistic case. The comparison between the relativistic and the Euclidean case will be made elsewhere.

Elsewhere also we hope to be able to discuss the interesting relations between relativistic crystallographic groups and Lie groups. In the Euclidean case such relations play an important rôle in the classification of complex semi-simple Lie algebras and in the representation theory of linear Lie groups[10–14].

2. *Relativistic lattices and crystallographic Lorentz transformations.* Given the two-dimensional Minkowskian vector space V with metric tensor

$$g(B) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(we put for the velocity of light: $c = 1$), and orthonormal basis $B = (e_1, e_2)$.

We consider a basis $B' = (e_1', e_2') = BS$ defined by

$$\begin{aligned} e_1' &= s_{11}e_1 + s_{21}e_2 \\ e_2' &= s_{12}e_1 + s_{22}e_2 \end{aligned} \qquad \text{with} \qquad S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \in GL(2, R) \tag{2.1}$$

and the lattice generated by B'

$$\Lambda(B') = \{z_1e_1' + z_2e_2' \mid z_1, z_2 \in Z\}.$$

The lattice $\Lambda(B')$ is a set of vectors in V. If T is a unimodular transformation, $T \in GL(2, Z)$, then $\Lambda(B'T) = \Lambda(B')$. We shall write simply Λ when it is not necessary to specify the basis.

$GL(2, R)$ and $GL(2, Z)$ are the groups of real and integer non-singular two-by-two matrices, respectively. The metric tensor corresponding to the basis B' is given by

$$g(B') = S^t g(B)\, S. \tag{2.2}$$

One finds:

$$g(B') = \begin{pmatrix} s_{11}^2 - s_{21}^2 & s_{11}s_{12} - s_{21}s_{22} \\ s_{11}s_{12} - s_{21}s_{22} & s_{12}^2 - s_{22}^2 \end{pmatrix} \stackrel{\text{Def}}{=} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \stackrel{\text{Def}}{=} (a, b, c) \tag{2.3}$$

and of course $\det g(B') < 0$, which means that

$$d \stackrel{\text{Def}}{=} b^2 - 4ac > 0. \tag{2.4}$$

The quantity d is called the discriminant of (a, b, c).

Those Lorentz transformation are crystallographic that are unimodular with respect to a certain basis B', *i.e.*:

$$L' = S^{-1}LS, \qquad \text{with} \qquad L \in O(1, 1), \qquad L' \in GL(2, Z) \tag{2.5}$$

for some $S \in GL(2, R)$. Since by definition: $L^t g(B)\, L = g(B)$ it follows that

$$L'^t g(B')\, L' = g(B'). \tag{2.6}$$

The unimodular character of L' ensures the invariance of the lattice Λ generated by B' under this Lorentz transformation, so that we may write:

$$L'\Lambda = \Lambda. \tag{2.7}$$

Any set of transformations $\{L_1', L_2', \ldots, L_\nu'\}$ satisfying the above three relations generates a relativistic point group K which is a subgroup of $GL(2, Z)$ and leaves Λ invariant. The largest point group H leaving a given lattice invariant is called its holohedry.

Two metric tensors $g(B_1)$ and $g(B_2)$ are arithmetically equivalent $g(B_1) \overset{a}{\sim} g(B_2)$ and belong to the same arithmetic class if

$$g(B_1) = S^t g(B_2)\, S \qquad \text{for some} \qquad S \in GL(2, Z), \tag{2.8}$$

whereas the arithmetic equivalence of point groups is given by

$$K_1 \overset{a}{\sim} K_2 \qquad \text{if} \qquad S^{-1}K_1S = K_2 \qquad \text{for some} \qquad S \in GL(2, Z). \tag{2.9}$$

Relation (2.9) means that K_1 and K_2 are two conjugate subgroups of $GL(2, Z)$.

To a given lattice there corresponds the whole set $\{B'S \mid S \in GL(2, Z)\}$ of its possible bases, and thus an arithmetic class $\{g(B')\}$ of metric tensors and also an arithmetic class $\{K\}$ of point groups. The concept of Bravais classes arises from the fact that whereas a lattice Λ determines the arithmetic class of its holohedry $\{H\}$, an arithmetic class determines a Bravais class of lattices[15]. Two lattices belong to the same Bravais class if they have arithmetically equivalent holohedries:

$$\Lambda_1 \overset{B}{\sim} \Lambda_2 \qquad \text{if} \qquad H_1 \overset{a}{\sim} H_2. \tag{2.10}$$

As the dimension of the space is even, it is possible to consider lattices with orientation, and oriented Bravais classes. In this case conjugation occurs with respect to an element of $SL(2, Z)$, the subgroup of proper elements of $GL(2, Z)$[15]. In this paper we shall consider oriented lattices. If the holohedry of such a lattice is a subgroup of $SL(2, Z)$, then its arithmetic class splits into two proper arithmetic classes and two enantiomorphic lattices Λ and Λ' belong to the same (non-oriented) Bravais class, where Λ' arises from Λ by changing orientation. If the holohedry of Λ is not a subgroup of $SL(2, Z)$ then $\Lambda = \Lambda'$ and the lattice is called ambiguous. Further on we shall discuss this in greater detail and in a more general frame.

The holohedry H of a lattice is determined by the metric tensor of any lattice basis B', because the relation

$$L'^{t}g(B')\,L' = g(B') \qquad \text{with} \qquad L' \in GL(2, Z) \tag{2.11}$$

is a necessary and sufficient condition for L' to belong to H. But a metric tensor does not determine a single lattice, because it simply fixes the length and the relative position of the two basis vectors.

Thus bases B_1 and B_2 (and therefore lattices Λ_1 and Λ_2) obtained one from the other by a Lorentz transformation: $B_2 = B_1 L$,

$$g(B_2) = L^{t}g(B_1)\,L = g(B_1) \qquad \text{with} \qquad L \in GL(2, Z) \tag{2.12}$$

determine the same metric tensor.

One says that B_1 and B_2, and the respective lattices Λ_1 and Λ_2, are relativistic-geometrically equivalent ($B_1 \overset{\text{rg}}{\sim} B_2$; $\Lambda_1 \overset{\text{rg}}{\sim} \Lambda_2$). Two lattices then belong to the same relativistic geometric class if they differ simply in the frame of reference. From (2.11) and (2.12) one arrives at the conclusion that relativistic-geometrically equivalent lattices belong to the same Bravais class.

Consider now any lattice. It contains a space-like vector which can be taken as first basis vector e_1'. Then $\|e_1'\|^2 = a > 0$. In other words a lattice determines an arithmetic class of metric tensors; in this class there is always a metric tensor $g(B') = (a, b, c)$ with $a > 0$. This means that if a is negative, one has to go over to an arithmetically equivalent metric tensor in order to meet the requirement.

By taking another representative of the Bravais class of the lattice considered above, it is always possible to have the first basis vector e_1' in the space direction e_1. This fixes the basis B' (and the lattice Λ) according to:

$$\begin{aligned} e_1' &= \sqrt{a}\, e_1; \\ e_2' &= \frac{b}{2\sqrt{a}}\, e_1 + \frac{d}{2\sqrt{a}}\, e_2. \end{aligned} \tag{2.13}$$

By this procedure, to each lattice Λ a metric tensor (a, b, c) with $a > 0$ is associated. Henceforth (if nothing else is explicitly stated) we shall always adhere to the choice (2.13).

The condition (2.5) for L' to be a proper Lorentz transformation leaving Λ invariant becomes:

$$S^{-1}LS = L' = \begin{pmatrix} \frac{1}{2}(n - bu) & -cu \\ au & \frac{1}{2}(n + bu) \end{pmatrix} \in GL(2, Z) \tag{2.14}$$

where

$$L = \pm \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}, \qquad S = \begin{pmatrix} \sqrt{a} & \dfrac{b}{2\sqrt{a}} \\ 0 & \dfrac{\sqrt{d}}{2\sqrt{a}} \end{pmatrix},$$

with $n = \pm 2 \cosh \chi$ and $u = \pm \sqrt{\{(n^2 - 4)/d\}}$ with sign u = sign χ. This means that n and u are solutions of the equation

$$n^2 - du^2 = 4 \tag{2.15}$$

expressing the condition det $L' = +1$. One easily verifies that in fact L' of (2.14) leaves invariant $g(B') = (a, b, c)$. Let us remark that the only transformations L' of finite order are those associated with the trivial solutions $u = 0$ and $n = \pm 2$, *i.e.* the identity E and the total inversion $-E$.

Improper Lorentz transformations are of the form

$$M = \pm \begin{pmatrix} \cosh \chi & \sinh \chi \\ -\sinh \chi & -\cosh \chi \end{pmatrix},$$

therefore traceless. Referred to the lattice basis B' of (2.13), they become:

$$S^{-1}MS = M' = \begin{pmatrix} \frac{1}{2}(n + bu) & -cu + \dfrac{b}{2a}(n + bu) \\ -au & -\frac{1}{2}(n + bu) \end{pmatrix}. \tag{2.16}$$

Note that M' is always of order two and can be written as:

$$M' = M'_0 L', \qquad M'_0 = \begin{pmatrix} 1 & \dfrac{b}{a} \\ 0 & -1 \end{pmatrix} \tag{2.17}$$

with L' as in (2.14). Therefore if a divides b and L' leaves Λ invariant, then so does M', and Λ is ambiguous.

A metric tensor $g(B') = (a, b, c)$ is called integral when a, b, c are rational integers. If furthermore a, b, c are relatively prime, then $g(B')$ is called primitive. The lattices corresponding by (2.13) to these tensors also are called integral or primitive lattices. We now formulate a property that is of fundamental importance in relativistic crystallography.

Proposition 1. Each Bravais class of lattices that have holohedries of infinite order contains a primitive lattice; at most two primitive lattices are contained in such a class.

Proof. We first show that there is a primitive lattice. According to the hypothesis, there is in the Bravais class a lattice Λ, with $g(B') = (a, b, c)$ with some $a, b, c \in R$, left invariant by a L', as in (2.14), with $u \neq 0$. We may suppose $u > 0$; otherwise we consider L'^{-1} instead of L'. Note that lattices differing only by a positive scaling factor are invariant with respect to the same point groups and belong to the same Bravais class. Therefore $g(uB') = (au, bu, cu)$ defines by (2.13) an integral lattice $u\Lambda$ that is in the same Bravais class as Λ. Consider the greatest positive divisor k of au, bu and cu. Then $u\Lambda/k$ is primitive and is in that same Bravais class. For $g(uB'/k)$ we write again (a, b, c) but now $a, b, c \in Z$ are relatively prime.

We now show that there are at most two primitive lattices in a same Bravais class. Let

$$L' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, Z)$$

be a proper Lorentz transformation of infinite order leaving invariant a primitive lattice with $g(B) = (a, b, c)$. Then by (2.14) one has

$$\gamma = au, \qquad \delta - \alpha = bu, \qquad -\beta = cu \qquad \text{and} \qquad \alpha + \delta = n \tag{2.18}$$

implying

$$|u| = gcd(\gamma, \delta - \alpha, -\beta) \in Z \qquad (u \neq 0), \tag{2.19}$$

i.e. u is the greatest common divisor of γ, $\delta - \alpha$ and $-\beta$. In order to have $a > 0$, one has to take:

$$\text{sign } u = \text{sign } \gamma. \tag{2.20}$$

By (2.18), (2.19) and (2.20), together with (2.13), a primitive lattice with metric tensor $g(B) = (a, b, c)$ is uniquely determined.

L' leaves invariant also $-g(B) = (-a, -b, -c)$ which defines an inverse lattice $\bar{\Lambda}$ of Λ (see section 4) and possibly $\Lambda \neq \bar{\Lambda}$.

Corollary 1.1. Every element $L' \in SL(2, Z)$ with $|\text{tr } L'| > 2$ is a crystallographic Lorentz transformation referred to the basis (2.13) of a primitive lattice.

Corollary 1.2. The general form of a proper crystallographic Lorentz transformation is:

$$L' = \begin{pmatrix} \frac{1}{2}(n - bu) & -cu \\ au & \frac{1}{2}(n + bu) \end{pmatrix} \tag{2.21a}$$

for any $a, b, c \in Z$ such that $gcd(a, b, c) = 1$ and $a > 0$, and for (n, u) any integral solution of the Pell (plus) equation:

$$n^2 - du^2 = 4, \qquad \text{where} \qquad d = b^2 - 4ac > 0. \tag{2.21b}$$

In this way we arrive at the theory of indefinite binary quadratic forms and of real quadratic fields, which of course are nothing else than other aspects of the same thing. The formulation of this matter in crystallographic terms is precisely the aim of this paper. From the theory of binary forms and quadratic fields we therefore mention only a minimum number of essential points.

Consider the indefinite quadratic form:

$$f(x, y) = ax^2 + bxy + cy^2 \overset{\text{Def}}{=} [a, b, c], \qquad \begin{matrix} \forall x, y \in Z \\ a, b, c \in R \end{matrix} \tag{2.22}$$

with discriminant

$$d = b^2 - 4ac > 0.$$

One says that the real numbers $f(x, y)$ are represented by the quadratic form $[a, b, c]$. Clearly, one may make a lattice Λ in the hyperbolic plane associated with the metric tensor $g(B') = (a, b, c)$ correspond to the set of numbers represented by $[a, b, c]$. The number $f(x, y)$ then expresses the distance between the origin and a lattice point P with coordinates x and y with respect to the basis B' of (2.13). We may say that the lattice Λ is represented by the quadratic form $[a, b, c]$. One has the following properties.

Proposition 2.

a) Two arithmetically equivalent quadratic forms (*i.e.* conjugated in $GL(2, Z)$) represent the same set of numbers.

b) To quadratic forms that are conjugated in $SL(2, Z)$ there corresponds one single (oriented) lattice.

c) To quadratic forms that are conjugated by an improper element of $GL(2, Z)$ there correspond two lattices, one of which is the mirror image of the other (with respect to the first basisvector). The two enantiomorphic lattices may coincide (ambiguous case).

Proof. See Bachmann[16]), pp. 100–117. Conjugate of quadratic forms occurs as for metric tensors after identification of $[a, b, c]$ with (a, b, c).

Suppose now

$$A^{\mathrm{t}} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}, \tag{2.23}$$

then any $A \in GL(2, R)$ satisfying (2.23) defines an automorphism of the quadratic form (a, b, c), and any $A \in GL(2, Z)$ satisfying (2.23) an automorph of the same form[17]). Comparison with (2.12) and (2.11) shows that Lorentz transformations are automorphisms, and the crystallographic ones automorphs of the quadratic form defined by the metric of the basis vectors (and *vice versa*). Actually, in order to obtain the relativistic Bravais classes one needs a generalization of these concepts, as is done in section 3. But before doing this, we add a few remarks on real quadratic fields[18,19]).

Consider the real quadratic field $Q(\sqrt{D})$ generated by $\sqrt{D}$ over the rationals Q, where D is a positive rational integer ($D > 0$). We assume D is not a square, but it may have square divisors. The square-free part is noted by D_0. One has:

$$Q(\sqrt{D}) = Q(\sqrt{D_0}) \qquad \text{where} \qquad D = q^2 D_0, \qquad q \in Z. \tag{2.24}$$

Any element $\lambda \in Q(\sqrt{D})$ can be written as

$$\lambda = \frac{z_1 + z_2\sqrt{D}}{z_3}, \qquad \forall z_1, z_2, z_3 \in Z. \tag{2.25}$$

The uniquely defined element:

$$\lambda' = \frac{z_1 - z_2\sqrt{D}}{z_3} \tag{2.26}$$

is called the conjugate of λ.

A norm N and a trace S are defined by

$$N(\lambda) \stackrel{\text{Def}}{=} \lambda\lambda' \qquad \text{and} \qquad S(\lambda) \stackrel{\text{Def}}{=} \lambda + \lambda' \tag{2.27}$$

with properties

$$\begin{aligned} &N(\lambda) = N(\lambda'); \quad N(\lambda_1\lambda_2) = N(\lambda_1)\,N(\lambda_2); \quad N(\lambda) = 0 \Leftrightarrow \lambda = 0 \\ &S(\lambda_1 + \lambda_2) = S(\lambda_1) + S(\lambda_2); \quad S(k\lambda) = kS(\lambda) \quad \text{for} \quad k \in Q. \end{aligned} \tag{2.28}$$

Quadratic integers ν are defined as those elements of $Q(\sqrt{D})$ which have rational integral norm and trace. One shows[18]) that they form a ring and are of the form :

$$\nu = \frac{z_1 + z_2\sqrt{d}}{2}, \qquad z_1, z_2 \in Z \tag{2.29}$$

where d is the discriminant of $Q(\sqrt{D})$ given by

$$d = \begin{cases} 4D & \text{if} \quad D \equiv 2 \text{ or } 3 \pmod 4; \\ D & \text{if} \quad D \equiv 1 \pmod 4. \end{cases} \tag{2.30}$$

The quadratic field $Q(\sqrt{D})$ will also be denoted by $Q(\sqrt{d})$. The quadratic integers of $Q(\sqrt{D})$ that have an integer inverse are called units of the quadratic field. They have norm ± 1 and form a group under multiplication, called the group of units. Accordingly units are given by

$$\varepsilon = \frac{n + u\sqrt{d}}{2} \tag{2.31a}$$

where $n, u \in Z$ are solution of the (plus, minus) Pell equation:

$$n^2 - du^2 = \pm 4. \tag{2.31b}$$

It is now possible to indicate the relation between quadratic fields and two-dimensional relativistic crystallography.

Relativistic Bravais lattices having holohedries of infinite order can be brought in correspondence with Z-modules $\mathscr{M}$ of quadratic integers. Consider the Bravais lattice Λ defined by the basis B' of (2.13) with primitive metric tensor $g(B') = (a, b, c)$, $a > 0$ and discriminant $d = b^2 - 4ac > 0$. The correspondence is then indicated in the following table (2.32).

TABLE (2.32)

Correspondence between relativistic Bravais lattices and Z-modules in real quadratic fields

	Minkowskian vector space V	Real quadratic field $Q(\sqrt{D})$	Discriminant
basis	$e_1, \sqrt{D}\, e_2$	$1, \omega \overset{\text{Def}}{=} \sqrt{D}$	$d = 4D$
lattice or module	$2\sqrt{a}\, \Lambda$	$\mathscr{M} = [2a, b + \omega]$	$d = 4q^2 D_0$
basis	$e_1, \frac{1}{2}(qe_1 + \sqrt{D}\, e_2)$	$1, \omega \overset{\text{Def}}{=} \frac{1}{2}(q + \sqrt{D})$	$d = D$
lattice or module	$\sqrt{a}\, \Lambda$	$\mathscr{M} = \left[a, \frac{b-q}{2} + \omega\right]$	$d = q^2 D_0$

In the case $d = D$ one has $a \equiv b \pmod 2 \equiv q \pmod 2$. The correspondence is such that if $\lambda \in Q(\sqrt{D})$ corresponds to $l \in V$, then

$$N(\lambda) = \|l\|^2; \qquad S(\lambda) = 2l \cdot e_1. \tag{2.33}$$

Note that $2\sqrt{a}\, \Lambda$, as well as $\sqrt{a}\, \Lambda$ are lattices in the same Bravais class as Λ.

Comparison of (2.31) with (2.21) shows that for a given primitive lattice (a, b, c), crystallographic proper Lorentz transformations

$$L = \begin{pmatrix} \frac{1}{2}(n - bu) & -cu \\ au & \frac{1}{2}(n + bu) \end{pmatrix}$$

are in one-to-one correspondence with the positive units ($N(\varepsilon) = +1$) of $Q(\sqrt{D})$ given by:

$$\varepsilon = \tfrac{1}{2}(n - u\sqrt{d}).$$

In this way we see that proper crystallographic Lorentz transformations and relativistic Bravais lattices (in the primitive case) appear as elements of one and the same quadratic field. The interpretation of negative units ($N(\varepsilon) = -1$) will be indicated in the following section.

3. *The g-automorphs.* We first consider the case of real (a, b, c), later on that of primitive (a, b, c), and we identify quadratic forms with corresponding metric tensors.

Definition. An automorphism X of a real indefinite metric tensor $g(B) = (a, b, c)$ is an element of $GL(2, R)$ such that

$$X^t g(B)\, X = g(B), \qquad X \in GL(2, R). \tag{3.1}$$

Definition. A negautomorphism X of a real indefinite metric tensor $g(B)$ is an element of $GL(2, R)$ such that:

$$X^t g(B)\, X = -g(B), \qquad X \in GL(2, R). \tag{3.2}$$

If no distinction is made between automorphism and negautomorphism we speak of generalized automorphisms (or simply of g-automorphisms). In the case of $g(B) = (1, 0, -1)$ we have the following possible general parametrization of g-automorphisms for $\phi \in R$:

$$\pm L(\phi) = \pm\begin{pmatrix}\cosh\phi & \sinh\phi\\ \sinh\phi & \cosh\phi\end{pmatrix} \quad \text{proper automorphism (Lorentz transformation);} \tag{3.3a}$$

$$\pm M(\phi) = \pm\begin{pmatrix}\cosh\phi & \sinh\phi\\ -\sinh\phi & -\cosh\phi\end{pmatrix} \quad \text{improper automorphism (Lorentz mirror);} \tag{3.3b}$$

$$\pm A(\phi) = \pm\begin{pmatrix}\sinh\phi & \cosh\phi\\ \cosh\phi & \sinh\phi\end{pmatrix} \quad \text{improper negautomorphism;} \tag{3.3c}$$

$$\pm N(\phi) = \pm\begin{pmatrix}\sinh\phi & \cosh\phi\\ -\cosh\phi & -\sinh\phi\end{pmatrix} \quad \text{proper negautomorphism.} \tag{3.3d}$$

In what follows we shall use the letters L, M, A, N, for the above g-automorphisms.

One easily verifies the following relations:

$$\begin{aligned}
&A(\phi_1)\ A(\phi_2) = A(\phi_2)\ A(\phi_1) = L(\phi_1+\phi_2);\\
&L(\phi_1)\ L(\phi_2) = L(\phi_2)\ L(\phi_1) = L(\phi_1+\phi_2);\\
&M(\phi_1)\ M(\phi_2) = (M(\phi_2)\ M(\phi_1))^{-1} = L(\phi_2-\phi_1);\\
&N(\phi_1)\ N(\phi_2) = (N(\phi_2)\ N(\phi_1))^{-1} = -L(\phi_2-\phi_1);\\
&A^{-1}(\phi) = A(-\phi), \qquad L^{-1}(\phi) = L(-\phi);\\
&M^{-1}(\phi) = M(\phi), \quad \text{therefore} \quad M^2(\phi) = 1;\\
&N^{-1}(\phi) = -N(\phi), \quad \text{therefore} \quad N^2(\phi) = -1.
\end{aligned} \tag{3.4}$$

The result of conjugation of a g-automorphism X_1 by another g-automorphism X_2, *i.e.* the element $X_2^{-1}X_1X_2$ is indicated in the following table.

TABLE (3.5)

Conjugations of g-automorphisms

g-automorphism / conjugated by	L_2	A_2	M_2	N_2
L_1	L_1	L_1	L_1^{-1}	L_1^{-1}
A_1	A_1	A_1	$-A_1^{-1}$	$-A_1^{-1}$
M_1	$M_1L_2^2$	$-M_1A_2^2$	$M_2M_1M_2$	$-M_1$
N_1	$N_1L_2^2$	$-N_1A_2^2$	$-N_1$	$-N_2N_1N_2$

We now go over to primitive metric tensors.

Definition. An automorph Y of a primitive metric tensor $g(B) = (a, b, c)$ with $a > 0$, is an element of $GL(2, Z)$ such that

$$Y^t g(B)\, Y = g(B), \qquad Y \in GL(2, Z). \tag{3.6}$$

Definition. A negautomorph Y of a primitive metric tensor $g(B)$ as above is an element of $GL(2, Z)$ such that:

$$Y^t g(B)\, Y = -g(B), \qquad Y \in GL(2, Z). \tag{3.7}$$

Elements of $GL(2, Z)$ satisfying (3.6) or (3.7) are called g-automorphs of $g(B)$.

Before discussing the basic properties of automorphs we introduce arithmetic functions $p_k(n)$ defined by the recurrence relation:

$$p_{k+1}(n) = n p_k(n) - p_{k-1}(n), \qquad \forall n, k \in Z \tag{3.8a}$$

and the initial values:

$$p_0(n) = 0, \qquad p_1(n) = 1, \qquad \forall n \in Z. \tag{3.8b}$$

Furthermore we define:

$$\Delta p_k(n) \overset{\text{Def}}{=} p_{k+1}(n) - p_{k-1}(n), \qquad \forall n, k \in Z \tag{3.9}$$

also obeying the recurrence relation (3.8a) but now with the initial values:

$$\Delta p_0(n) = 2 \quad \text{and} \quad \Delta p_1(n) = n, \qquad \forall n \in Z. \tag{3.10}$$

The detailed properties of these functions will be discussed elsewhere. Let us here mention only a useful parametrization in term of trigonometric and hyperbolic functions.

For any $n \in Z$, consider $\phi \in R$ with $\phi > 0$ such that

$$n = \begin{cases} (\operatorname{sign} n)\, 2 \cosh \phi & \text{if} \quad |n| \geq 2; \\ 2 \cos \phi & \text{if} \quad |n| \leq 2. \end{cases} \tag{3.11}$$

Then

$$\text{for} \quad |n| \leq 2 \begin{cases} 2 \cos k\phi = \Delta p_k(n); \\ 2 \sin k\phi = \sqrt{(4 - n^2)}\, p_k(n) \end{cases} \tag{3.12a}$$

and

$$\text{for} \quad n \geq 2 \begin{cases} 2 \cosh k\phi = \Delta p_k(n); \\ 2 \sinh k\phi = \sqrt{(n^2 - 4)}\, p_k(n), \end{cases} \tag{3.12b}$$

with $p_k(-n) = (-1)^{k+1} p_k(n)$ and $\Delta p_k(-n) = (-1)^k \Delta p_k(n)$.

Note that $\Delta p_k(n)$ and $p_k(n)$ are solutions of the Pell (plus) equation of discriminant $d = n^2 - 4$:

$$\Delta p_k^2(n) - (n^2 - 4)\, p_k^2(n) = 4. \tag{3.13}$$

Proposition 3. A proper automorph L of a primitive $g(B) = (a, b, c)$

can always be written as:

$$L = \pm L_0^k = \pm \begin{pmatrix} \frac{1}{2}(n_1 - bu_1) & -cu \\ au_1 & \frac{1}{2}(n_1 + bu_1) \end{pmatrix}^k = \pm \begin{pmatrix} \frac{1}{2}(n_k - bu_k) & -cu_k \\ au_k & \frac{1}{2}(n_k + bu_k) \end{pmatrix}, \tag{3.14}$$

where (n_1, u_1) is the least positive solution of the Pell (plus) equation of discriminant $d = b^2 - 4ac > 0$:

$$n^2 - du^2 = 4$$

and

$$n_k = \Delta p_k(n_1), \qquad u_k = u_1 p_k(n_1). \tag{3.15}$$

We call L_0 the fundamental Lorentz transformation (of $g(B)$).

Proof. For any given primitive $g(B) = (a, b, c)$ there is a one-to-one correspondence between positive units ε of the quadratic field $Q(\sqrt{d})$, where $d = b^2 - 4ac$, and the proper automorphs L of $g(B)$:

$$L = \begin{pmatrix} \frac{1}{2}(n - bu) & -cu \\ au & \frac{1}{2}(n + bu) \end{pmatrix} \Leftrightarrow \varepsilon = \tfrac{1}{2}(n + u\sqrt{d}). \tag{3.16}$$

We now show that L^k is given by

$$L^k = \begin{pmatrix} \frac{1}{2}(n_k - bu_k) & -cu_k \\ au_k & \frac{1}{2}(n_k + bu_k) \end{pmatrix}$$

where

$$n_k = \Delta p_k(n) \quad \text{and} \quad u_k = up_k(n).$$

For $k = 1$ the statement is trivially true. Suppose it is true for $k - 1$, then for L^k one finds the above expression with

$$n_k = \tfrac{1}{2}(n_{k-1}\, n + u_{k-1}\, ud),$$
$$u_k = \tfrac{1}{2}(u_{k-1}\, n + n_{k-1}\, u),$$

where, according to the induction hypothesis:

$$n^2 - du^2 = 4, \quad n_{k-1}^2 - du_{k-1}^2 = 4, \quad n_{k-1} = \Delta p_{k-1}(n) \quad \text{and}$$
$$u_{k-1} = up_{k-1}(n).$$

Using the relations:

$$p_{k_1+k_2}(n) = \tfrac{1}{2}(p_{k_1}(n)\, \Delta p_{k_2}(n) + \Delta p_{k_1}(n)\, p_{k_2}(n)),$$
$$\Delta p_{k_1+k_2}(n) = \tfrac{1}{2}(\Delta p_{k_1}(n)\, \Delta p_{k_2}(n) + (n^2 - 4)\, p_{k_1}(n)\, p_{k_2}(n)) =$$
$$= \tfrac{1}{2}(\Delta p_{k_1}(n)\, \Delta p_{k_2}(n) + p_{k_1}(n)\, p_{k_2}(n)\, ud),$$

which are a simple consequence of (3.12), one finds:

$$n_k = \Delta p_k(n) \quad \text{and} \quad u_k = up_k(n).$$

In exactly the same way one proves that

$$\varepsilon^k = \tfrac{1}{2}(n_k + u_k\sqrt{d}).$$

It follows that L^k corresponds to ε^k.

Any positive unit of $Q(\sqrt{d})$ can be written as $\varepsilon = \pm\varepsilon_1^k$, where ε_1 is the fundamental (positive) unit obtained from the least positive solution (u_1, n_1) of the Pell equation (see *e.g.* ref. 18), so that any proper automorph of $g(B)$ is given by:

$$L = \pm L_0^k$$

with L_0 the fundamental Lorentz transformation (of $g(B)$).

Note that in our case $|n_1| > 2$ if d is not a square.

Corollary 3.1. The crystallographic Lorentz transformations L_0 and $L = \pm L_0^k$ of (a, b, c) are in correspondence uniquely to $\varepsilon_1 = \frac{1}{2}(n_1 + u_1\sqrt{d})$ the fundamental (positive) unit of $Q(\sqrt{d})$, and to $\varepsilon = \pm\varepsilon_1^k$, respectively.

Corollary 3.2. An improper automorph M of a primitive $g(B) = (a, b, c)$ can always be written as

$$M = \pm M_0 L_0^k = \pm \begin{pmatrix} \frac{1}{2}(n_k + bu_k) & -cu_k + \dfrac{b}{2a}(n_k + bu_k) \\ -au_k & -\frac{1}{2}(n_k + bu_k) \end{pmatrix} \tag{3.17}$$

where

$$M_0 = \begin{pmatrix} 1 & \dfrac{b}{a} \\ 0 & -1 \end{pmatrix}$$

and n_k, u_k as above.

Proof. Consider (2.16) and (2.17).

The negautomorphs can be treated in the same way, with an important restriction, however; we have to consider only those primitive $g(B)$ for which there are negautomorphs. The explicit conditions therefore are discussed further on. Let us, for the time being, merely underline the fact that there are (oriented) lattices that admit proper negautomorphs but no improper ones. (See section 4.)

We first introduce new arithmetic functions $q_k(m)$ defined by the recurrent relation:

$$q_{k+1}(m) = mq_k(m) + q_{k-1}(m), \qquad \forall m, k \in Z \tag{3.18a}$$

and the initial values:

$$q_0(m) = 0, \qquad q_1(m) = 1, \qquad \forall m \in Z. \tag{3.18b}$$

Furthermore we define

$$\Delta q_k(m) \overset{\text{Def}}{=} q_{k+1}(m) + q_{k-1}(m), \qquad \forall k, m \in Z \tag{3.19}$$

also obeying the recurrent relation (3.18a) but with the initial values:

$$\Delta q_0(m) = 2 \qquad \text{and} \qquad \Delta q_1(m) = m, \qquad \forall m \in Z. \tag{3.20}$$

Note that $q_k(1)$ are the numbers of Fibonacci and $\Delta q_k(1)$ those of Lucas. These arithmetic functions too can be parametrized in terms of hyperbolic functions.

Consider for any $m \in Z$ a $\phi \in R$ such that

$$2 \sinh \phi = m. \tag{3.21}$$

Then for any $k \in Z$:

$$\begin{cases} 2 \sinh 2k\phi = \sqrt{(m^2 + 4)}\, q_{2k}(m), \\ 2 \cosh 2k\phi = \Delta q_{2k}(m) \end{cases}$$

and

$$\begin{cases} 2 \sinh (2k + 1)\, \phi = \Delta q_{2k+1}(m), \\ 2 \cosh (2k + 1)\, \phi = \sqrt{(m^2 + 4)}\, q_{2k+1}(m) \end{cases} \tag{3.22}$$

where $\Delta q_k(m)$ and $q_k(m)$ are solutions of the Pell equation (plus, minus) of discriminant $d = m^2 + 4$:

$$\Delta q_k^2(m) - (m^2 + 4)\, q_k^2(m) = (-1)^k\, 4. \tag{3.23}$$

Proposition 4. Suppose $g(B') = (a, b, c)$, $a > 0$, a primitive metric tensor admitting as in (3.7) an improper negautomorph A. Then A can always be written as:

$$A = \pm A_0^k = \pm \begin{pmatrix} \frac{1}{2}(m_1 - bv_1) & -cv_1 \\ av_1 & \frac{1}{2}(m_1 + bv_1) \end{pmatrix}^k = \pm \begin{pmatrix} \frac{1}{2}(m_k - bv_k) & -cv_k \\ av_k & \frac{1}{2}(m_k + bv_k) \end{pmatrix} \tag{3.24}$$

with $k \equiv 1 \pmod 2$ and (m_1, v_1) the least positive solution of the Pell (minus) equation of discriminant $d = b^2 - 4ac > 0$:

$$m^2 - dv^2 = -4, \tag{3.25a}$$

furthermore

$$m_k = \Delta q_k(m_1), \qquad v_k = v_1 q_k(m_1). \tag{3.25b}$$

Proof. One may regard an improper negautomorph A of $g(B')$ as an improper negautomorphism of $(1, 0, -1)$, which becomes unimodular when referred to the basis B' of (2.13). Therefore, using (3.3c) one gets:

$$A = \pm \begin{pmatrix} \dfrac{1}{\sqrt{a}} & -\dfrac{b}{\sqrt{ad}} \\ 0 & \dfrac{2\sqrt{a}}{\sqrt{d}} \end{pmatrix} \begin{pmatrix} \sinh \chi & \cosh \chi \\ \cosh \chi & \sinh \chi \end{pmatrix} \begin{pmatrix} \sqrt{a} & \dfrac{b}{2\sqrt{a}} \\ 0 & \dfrac{\sqrt{d}}{2\sqrt{a}} \end{pmatrix} =$$

$$= \pm \begin{pmatrix} \frac{1}{2}(m - bv) & -cv \\ av & \frac{1}{2}(m + bv) \end{pmatrix} \tag{3.26}$$

where $m = 2 \sinh \chi \in Z$, and $v = \sqrt{\{(m^2 + 4)/d\}} \in Z$. (m, v) are integral solutions of the Pell (minus) equation:

$$m^2 - dv^2 = -4.$$

Therefore $\varepsilon = \pm\frac{1}{2}(m + v\sqrt{d})$ is a negative unit of $Q(\sqrt{d})$ and is uniquely determined by A. Let A_0 be the improper negautomorph that corresponds to the fundamental (negative) unit ε_1 of $Q(\sqrt{d})$ given by: $\varepsilon_1 = \frac{1}{2}(m_1 + v_1\sqrt{d})$. Then $\varepsilon = \pm\varepsilon_1^k$ for some $k \equiv 1 \pmod 2$ (see *e.g.* ref. 18, p. 303) and correspondingly $A = \pm A_0^k$. Defining now ϕ by:

$$m_1 = 2 \sinh \phi \tag{3.27}$$

it follows that $2 \sinh k\phi = m_k = \Delta q_k(m_1)$ and $v_k = v_1 q_k(m_1)$, then, is a simple consequence of (3.23).

Corollary 4.1. The improper negautomorphs of infinite order $A = \pm A_0^{2k+1}$ of $g(B) = (a, b, c)$ are in one-to-one correspondence with the negative units $\varepsilon = \pm\varepsilon_1^{2k+1}$ of $Q(\sqrt{d})$, for $k \in Z$.

Remark. The restriction in corrollary 4.1 to negautomorphs of infinite-order, *i.e.* to the case $m_1 > 0$, is due to the fact that in the case of negautomorph of finite order the correspondence with negative units breaks down. For $m = 0$ the Pell (minus) equation becomes $dv^2 = 4$ with positive solutions:

$$v = 2 \quad \text{for} \quad d = 1,$$

$$v = 1 \quad \text{for} \quad d = 4.$$

In both cases $Q(\sqrt{d}) = Q$ and the only units are ± 1, thus positive. For $d = 1$, no primitive metric tensor is possible (see ref. 17 theorem 75, p. 101), but for $d = 4$, the metric tensor $g(B) = (1, 0, -1)$ admits improper negautomorphs of finite order:

$$A = \pm A_0 = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.28}$$

Note also that the group of g-automorphs of a given primitive $g(B)$, is in general not isomorphic with the group of units of $Q(\sqrt{d})$ for d being the discriminant of $g(B)$. In fact, no correspondence has been formulated between units of $Q(\sqrt{d})$ and improper automorphs or proper negautomorphs of $g(B)$.

The one-to-one correspondence between units and g-automorphs can be broadly summarized as follows:

positive units $\leftrightarrow$ proper automorphs
(also of finite order),

negative units $\leftrightarrow$ improper negautomorphs
(only of infinite order).

A table indicating real quadratic fields $Q(\sqrt{D_0})$ having negative units can be found in ref. 19, pp. 271–274 for D_0 up to 97.

We now go over to proper negautomorphs. In the same way as in (3.26) and using (3.13) and (3.3d) one sees that the general form of a proper negautomorph of a primitive $g(B) = (a, b, c)$, if it exists, is given by:

$$N = \pm \begin{pmatrix} \frac{1}{2}(m - bv) & \frac{b}{2a}(m - bv) + cv \\ av & -\frac{1}{2}(m - bv) \end{pmatrix} \tag{3.29}$$

with (m, v) solutions of the Pell (minus) equation (3.25a). Note that these solutions need be integral (and therefore of the form (3.25b)). Comparing with (2.16) and (2.17) one finds that N can be written as:

$$N = M_0 A, \qquad \text{with} \qquad M_0 = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & -1 \end{pmatrix}. \tag{3.30}$$

M_0 is an improper automorphism and A an improper negautomorphism as in (3.26). The point is that only if a divides b, *i.e.* if $g(B)$ is ambiguous, are M_0 and A necessarily unimodular, *i.e.* (m, v) are integral solutions of (3.25). In general this is not the case. As an example let us consider the lattice defined by $g(B) = (5, 11, -5)$. One verifies that $g(B)$ is not ambiguous and does not admit improper negautomorphs. In fact the discriminant is $d = D = 221$ and the fundamental unit of $Q(\sqrt{221})$ is $\varepsilon_1 = \frac{1}{2}(15 + \sqrt{221})$, thus of norm $+1$. But $g(B)$ does admit proper negautomorphs, namely, for example:

$$N_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.31}$$

which may indeed be obtained from (3.29) for $m = 11/5$ and $v = 1/5$, a non-integral solution of $m^2 - 221v^2 = -4$.

We think that in the case of enantiomorphic lattices, proper negautomorphs can always be put into the form (possibly after a suitable basis transformation):

$$N = N_0 L \tag{3.32}$$

where N_0 is as in (3.31) and L is an arbitrary proper automorph of $g(B)$. This is indeed the case in the example mentioned above, although in general, we have not been able to prove it. Clearly proper negautomorphs need further investigation.

4. *Classification of primitive lattices.* Let Λ be an oriented lattice defined by primitive $g(B) = (a, b, c)$, $a > 0$, according to (2.13). By Λ' we indicate

the mirror lattice of Λ given by:

$$\Lambda' = X\Lambda \qquad \text{with} \qquad X \in GL(2, Z) - SL(2, Z). \tag{4.1}$$

We know that if $\Lambda' = \Lambda$, the lattice is ambiguous, otherwise Λ and Λ' form an enantiomorphic pair.

We now define $\bar{\Lambda}$, the inverse of lattice Λ, as the lattice of $\{-g(B)\}_+$. Here we have to consider the proper arithmetic class of $-g(B) = (-a, -b, -c)$, because of the convention (2.13). We have the following correspondences

$$\Lambda \Leftrightarrow \{g(B)\}_+ \qquad \text{and} \qquad \bar{\Lambda} \Leftrightarrow \{-g(B)\}_+. \tag{4.2}$$

In the case where $\Lambda = \bar{\Lambda}$ we speak of a stable lattice, otherwise Λ and $\bar{\Lambda}$ are called unstable.

Proposition 5. A lattice Λ defined by primitive $g(B)$ according to (2.13) is stable if and only if it is left invariant by proper negautomorphs of $g(B)$.

Proof. Suppose $\Lambda = \bar{\Lambda}$, this means that $-g(B) \in \{g(B)\}_+$; thus there is a $N \in SL(2, Z)$ such that $N^t g(B) N = -g(B)$. Clearly N is a proper negautomorph. The converse is also true, as can be seen by reversing the arguments.

Corollary 5.1. A lattice Λ of a primitive $g(B) \neq (1, 0, -1)$ of discriminant $d > 0$, which is ambiguous and stable is left invariant by improper

TABLE (4.3)

Classification of relativistic primitive lattices

Transformations leaving Λ invariant		Characterization of the lattice		Type
improper automorphs $\Lambda = \Lambda'$	negautomorphs $\Lambda = \bar{\Lambda}$	ambiguous stable negative	$\Lambda = \Lambda'$ $\Lambda = \bar{\Lambda}$ $\Lambda = \bar{\Lambda}'$	I
	automorphs only $\Lambda \neq \bar{\Lambda}$	ambiguous unstable positive	$\Lambda = \Lambda'$ $\Lambda \neq \bar{\Lambda}$ $\Lambda \neq \bar{\Lambda}'$	II
no improper automorphs $\Lambda \neq \Lambda'$	proper negautomorphs $\Lambda = \bar{\Lambda}$	enantiomorphic stable positive	$\Lambda \neq \Lambda'$ $\Lambda = \bar{\Lambda}$ $\Lambda \neq \bar{\Lambda}'$	III
	improper negautomorphs $\Lambda = \bar{\Lambda}'$	enantiomorphic unstable negative	$\Lambda \neq \Lambda'$ $\Lambda \neq \bar{\Lambda}$ $\Lambda = \bar{\Lambda}'$	IV
	proper automorphs only (no negautomorphs)	enantiomorphic unstable positive	$\Lambda \neq \Lambda'$ $\Lambda \neq \bar{\Lambda}$ $\Lambda \neq \bar{\Lambda}'$	V

negautomorphs of $g(B)$. The real quadratic field $Q(\sqrt{d})$ has a negative fundamental unit.

We call such a lattice with $(\Lambda)' = (\bar{\Lambda}') = \bar{\Lambda}'$ a negative lattice. If $\Lambda \neq \bar{\Lambda}'$ the lattice is said to be positive. In this case there are no improper negautomorphs of $g(B)$ and the units of $Q(\sqrt{d})$ are all positive.

Using these various definitions relativistic primitive lattices can be divided into five different types as indicated in the table (4.3).

Explicit examples are indicated in the appendix showing that lattices of the five different types effectively occur.

There are infinitely many primitive lattices and it is therefore worthwhile to group some of them in families of lattices having some peculiar properties.

Given a value d of the discriminant, there is always a principal quadratic form with this discriminant (see *e.g.* ref. 16, p. 119). Accordingly we call principal lattices P_d, those lattices which are defined by the correspondence

$$P_d \Leftrightarrow \begin{cases} \left(1, 1, \dfrac{1-d}{4}\right) & \text{if} \quad d \equiv 1 \pmod 4, \qquad (4.4a) \\ \left(1, 0, -\dfrac{d}{4}\right) & \text{if} \quad d \equiv 0 \pmod 4. \qquad (4.4b) \end{cases}$$

Principal lattices are always ambiguous. For the inverse principal lattices $\bar{P}_d$ one finds:

$$\bar{P}_d \Leftrightarrow \begin{cases} \left(\dfrac{d-1}{4}, 1, -1\right) & \text{if} \quad d \equiv 1 \pmod 4, \qquad (4.5a) \\ \left(\dfrac{d}{4}, 0, -1\right) & \text{if} \quad d \equiv 0 \pmod 4. \qquad (4.5b) \end{cases}$$

The concept of natural lattice plays an important rôle in crystallography, even if not yet generally recognized. The name "natural" has been suggested by W. Opechowski.

In order to explain the idea, consider a g-automorph X of a lattice Λ. One has

$$\forall x \in \Lambda \qquad Xx = y \in \Lambda. \tag{4.6}$$

In general (x, y) is not a basis of Λ but only of a sublattice. Suppose now that there is an element $e_1' \in \Lambda$ and a g-automorph X of Λ such that (e_1', Xe_1') forms a basis of Λ. In this case we speak of a natural lattice generated by X. If X is an improper negautomorph

$$X \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we speak of a negative natural lattice; and if X is a proper automorph

$$X \neq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

we speak of a positive natural lattice.

Proposition 6. For any $n \in Z$, there are two positive natural lattices, M_n and its inverse $\bar{M}_n$. The lattice M_n is the principal lattice of discriminant $d = n^2 - 4$. One has the following correspondences

$$\left.\begin{aligned} M_{2\nu} &\Leftrightarrow (1, 0, 1 - \nu^2) \\ M_{2\nu+1} &\Leftrightarrow (1, 1, 1 - \nu(\nu + 1)) \\ \bar{M}_{2\nu} &\Leftrightarrow (\nu^2 - 1, 0, -1) \\ \bar{M}_{2\nu+1} &\Leftrightarrow (\nu(\nu + 1) - 1, 1, -1). \end{aligned}\right\} \quad \forall \nu \in Z \tag{4.7}$$

Proof. Consider a primitive lattice corresponding by (2.13) to (a, b, c). The general form of a proper automorph L being (2.21), the lattice generated by a vector $x = x_1 e_1' + x_2 e_2' \in \Lambda$ and its transformed $y = Lx = y_1 e_1' + y_2 e_2'$ with $y_1 = \frac{1}{2} x_1 (n - bu) - x_2 cu$ and $y_2 = x_1 au + \frac{1}{2} x_2 (n + bu)$ is a sublattice of index Δ in Λ, where

$$\Delta = (ax_1^2 + bx_1x_2 + cx_2^2)\, u. \tag{4.8}$$

For Λ natural: $\Delta = \pm 1$; consequently $u = \pm 1$. Without restriction we may suppose $u = 1$. Then for $\Delta = 1$

$$ax_1^2 + bx_1x_2 + cx_2^2 = 1 \qquad \text{and} \qquad d = n^2 - 4.$$

According to a theorem of Lagrange (see ref. 17, p. 111) there is in the proper arithmetic class of (a, b, c) a quadratic form (a', b', c') with $a' = 1$.

For $n = 2\nu + 1$ (odd n), b' is also odd and, indicating proper arithmetic equivalence by $\overset{+}{\sim}$, one has:

$$(a, b, c) \overset{+}{\sim} (1, b', c') \overset{+}{\sim} (1, 1, c'') = \left(1, 1, \frac{1 - d}{4}\right) = (1, 1, 1 - \nu(\nu + 1)).$$

For even n, $n = 2\nu$, b' is also even and, analogously, one finds:

$$(a, b, c) \overset{+}{\sim} (1, b', c') \overset{+}{\sim} (1, 0, c'') = \left(1, 0, -\frac{d}{4}\right) = (1, 0, 1 - \nu^2).$$

Suppose now $\Delta = -1$ and $ax_1^2 + bx_1x_2 + cx_2^2 = -1$. As above, in the proper arithmetic class there is an (a', b', c') with $c' = -1$, and one finds the inverse lattices $\bar{M}_n$ in the same way.

Corollary 6.1. For getting all positive natural lattices it is sufficient to consider all integers $n \geq 0$, since $M_{-n} = M_n$.

Proposition 7. The value $n = 3$ is the only one for which $\bar{M}_n = M_n$. The lattice M_3 is of type I.

Proof. For $n = 3$ the discriminant of M_3 is $d = 5$ and the fundamental unit of $Q(\sqrt{5})$ is negative[19]). Being ambiguous and negative, M_3 is stable.

Quadratic fields of discriminant $d = n^2 - 4$, for $|n| > 3$, have only positive units. To prove this, suppose that there are integral solutions of the corresponding Pell (minus) equation. Take the least positive one (m_1, v_1). It follows that the least positive solution of the Pell (plus) equation for same discriminant is given by

$$n_1 = m_1^2 + 2 \qquad \text{and} \qquad u_1 = m_1 v_1. \tag{4.9}$$

This can be seen by squaring the improper negautomorph A_0 of (3.24). But for natural lattices, $u_1 = 1$, and this implies $m_1 = v_1 = 1$, thus $n_1 = 3$, which is contrary to the hypothesis.

Proposition 8. For any $m \in Z$ there is a negative natural lattice N_m, which is the principal lattice of discriminant $d = m^2 + 4$. One has:

$$\begin{aligned} N_{2\mu} &\Leftrightarrow (1, 0, -(1 + \mu^2)) & \forall \mu \in Z, \quad \mu \neq 0, \\ N_{2\mu+1} &\Leftrightarrow (1, 1, 1 - \mu(\mu + 1)) & \forall \mu \in Z. \end{aligned} \tag{4.10}$$

Proof. The proof is obtained exactly as in proposition 7. One finds N_m generated by the improper negautomorph A (3.26) of a primitive (a, b, c) with $(m, v) = (m, 1)$, so that $d = m^2 + 4$. Applying Lagrange's theorem one arrives at the desired results.

Corollary 8.1. For getting all negative natural lattices, it is sufficient to consider integral positive m (since $N_{-m} = N_m$).

Corollary 8.2. Any negative natural lattice N_m is stable ($N_m = \overline{N}_m$).

Corollary 8.3. The only case where negative and positive natural lattices coincide is for $m = 1$ and $n = 3$, *i.e.*, $N_1 = M_3$.

We shall also mention two other properties that give a geometrical interpretation of the least positive solution of Pell's equation. We shall omit the proof as it can be derived straightforwardly.

Proposition 9. Suppose $d \in Z$ a given positive discriminant ($d \in Z$) and (n_1, u_1) the least positive solution of the corresponding Pell (plus) equation. The positive natural lattice M_{n_1} is a sublattice of index u_1 in the principal lattice P_d.

Proposition 10. Suppose d a given positive discriminant appearing in a Pell (minus) equation, which has non trivial solutions, and let (m_1, v_1) be the least positive one. The negative natural lattice M_{m_1} is a sublattice of index v_1 in the principal lattice P_d.

Quite generally, as all primitive lattices of same discriminant have equal surface of their elementary cell ($d = -4 \det g(B)$), there is an obvious generalization of the last two propositions.

5. *Some properties of ambiguous lattices.* The language of quadratic forms is a very convenient one for the matter discussed in this section and for referring to the literature. Only primitive quadratic forms are considered here, and in this section by equivalency (2.8).

We recall that a form (a, b, c) is ambiguous if a divides b. In this case the proper arithmetic class coincides with the arithmetic one (see ref. 17, p. 71): the lattice represented by the forms of the class is ambiguous and vice versa.

An ambiguous form is called rectangular if b/a is even and rhombic if b/a is odd. This nomenclature is understandable on the basis of the following proposition.

Proposition 11. A rectangular form is equivalent to one with $b = 0$; a rhombic form to one with $b = a$. More explicitly:

$$\begin{aligned} &(a, 2ka, c) \sim (a, 0, c - k^2a) \qquad \forall k \in Z, \\ &(a, (2k+1)\, a, c) \sim (a, a, c - k(k+1)\, a). \end{aligned} \tag{5.1}$$

Proof.

$$\begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}\begin{pmatrix} a & ka \\ ka & c \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & c - k^2a \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}\begin{pmatrix} a & \dfrac{2k+1}{2}\, a \\ \dfrac{2k+1}{2}\, a & c \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \dfrac{a}{2} \\ \dfrac{a}{2} & c - k(k+1)\, a \end{pmatrix}.$$

It is in general not easy to decide whether given a quadratic form (a, b, c) is equivalent to an ambiguous one or not, the number of equivalent forms being infinite. The standard approach is to go over to equivalent reduced forms (see ref. 18, p. 100). The conditions for a form (a, b, c) to be reduced are:

$$0 < \sqrt{d} - b < 2\,|a| < \sqrt{d} + b \qquad \text{and} \qquad 0 < b < \sqrt{d} \tag{5.2}$$

and imply also

$$0 < \sqrt{d} - b < 2\,|c| < \sqrt{d} + b. \tag{5.3}$$

The number of equivalent reduced forms is finite (ref. 17 theorem 79, p. 103). These can be ordered into a chain of equivalent reduced forms.

Proposition 12. A rectangular form is equivalent to a reduced rectangular form. The same is true if one replaces rectangular by rhombic.

Proof. We show first that any primitive $(A, 0, C)$ is equivalent to a reduced form $(a, 2ka, c)$. Suppose $|A| < |C|$ and take for k the largest rational integer satisfying the inequality: $0 < \sqrt{|C|} - |k|\sqrt{|A|}$ (d is not a square). The form $(A, 0, C)$ is equivalent to the reduced one $(A, 2kA, C + k^2A)$. If $|A| > |C|$ the same can be proved by considering the (equivalent) form $(C, 0, A)$, called the associate of $(A, 0, C)$.

In the case of a rhombic form, one verifies that any form $(a, |a|, c)$ with $-ac > 0$ is reduced. Now a rhombic form can always be given as $(\pm A, A, \pm C)$ with $A, C > 0$. The only non-reduced cases are (A, A, C) and $(-A, A, -C)$. The first form is equivalent to $(4C - A, A - 4C, C)$, the second one to $(A - 4C, A - 4C, C)$ which are both reduced rhombic forms.

We may distinguish between three sorts of ambiguous lattices:

(i) A lattice is rectangular (R) if any ambiguous form representing it is rectangular.

(ii) A lattice is rhombic (D) if any ambiguous form representing it is rhombic.

(iii) A lattice is mixed (RD) if it is represented by rhombic as well as by rectangular forms.

Corollary 12.1. In a chain of reduced equivalent forms representing an ambiguous lattice, there are always two ambiguous forms. If both of these are rectangular (or both rhombic) the lattice is rectangular (or rhombic). If one form is rectangular and the other is rhombic, the lattice is mixed (RD).

Proof. See ref. 17, p. 116 and use proposition 12. Let us remark that this classification of ambiguous lattices has a deeper meaning than one might be tempted to think initially. One can *e.g.* show that a RD-lattice cannot be of type I. This result will be proved and discussed in a subsequent paper[21]).

6. *Isotropic lattices.* A lattice is called isotropic if it contains isotropic lattice vectors. One may then always choose a lattice basis $B' = (e'_1, e'_2)$ with metric tensor $g(B') = (0, b, c)$ having a square discriminant $d = b^2$ $(b \in R)$.

In particular if the lattice is primitive, then its discriminant is a squared integer. This property is characteristic for isotropic primitive lattices.

Proposition 13. A primitive lattice is isotropic if and only if its discriminant is a squared integer.

Proof. We may suppose the lattice given by a primitive $g(B) = (a, b, c)$. The lattice is isotropic if and only if the corresponding quadratic form:

$$f(x, y) = ax^2 + bxy + cy^2, \qquad \forall x, y \in Z \tag{6.1}$$

represents zero over Z. The necessary and sufficient condition for (a, b, c) to represent zero over the rationals Q (*i.e.* for $x, y \in Q$) is that the discriminant d be a squared integer (see ref. 20, *p.* 395, theorem 10); conversely, if d is a squared integer, than (a, b, c) also represents zero over Z.

Proposition 14. The only proper automorphs of an isotropic lattice are E and $-E$.

Proof. Consider the metric tensor $g(B) = (0, b, c)$ of an isotropic lattice and its automorph

$$S \in \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, Z).$$

Then (for $b \neq 0$, because $d > 0$):

$$\beta = \gamma = 0, \qquad \alpha = \delta \quad \text{and} \quad \alpha\delta = 1. \tag{6.2}$$

Proposition 15. A primitive isotropic lattice given by $g(B) = (0, b, c)$ is:

(i) ambiguous if b divides $c^2 - 1$,

(ii) stable if b divides $c^2 + 1$,

(iii) negative if b divides 2.

Two of the three conditions imply the third one.

Proof. Let

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, Z)$$

and $g(B) = (0, b, c)$ with $c \neq 0$. If S is an improper automorph of $(0, b, c)$ then:

$$\alpha b + \gamma c = 0, \qquad \alpha = -\delta \quad \text{and} \quad \alpha\delta - \beta\gamma = -1.$$

This means

$$S = \pm \begin{pmatrix} c & \dfrac{c^2 - 1}{b} \\ -b & -c \end{pmatrix}. \tag{6.3}$$

If S is a proper negautomorph, then:

$$ab + \gamma c = 0, \qquad \alpha = -\delta \quad \text{and} \quad \alpha\delta - \beta\gamma = 1,$$

giving

$$S = \pm \begin{pmatrix} c & \dfrac{c^2 + 1}{b} \\ -b & -c \end{pmatrix}. \tag{6.4}$$

If now S is an improper negautomorph, then:

$$\gamma = 0, \qquad \beta b = (\alpha - \delta)\, c \quad \text{and} \quad \alpha\delta = -1,$$

i.e.:

$$S = \pm \begin{pmatrix} 1 & \frac{2c}{b} \\ 0 & -1 \end{pmatrix}. \tag{6.5}$$

As b and c are relatively prime, it follows that b divides 2. Clearly two of the three relations imply the third one.

If now $c = 0$, we have for the primitive lattice $g(B) = (0, 1, 0)$, and the isotropic lattice is ambiguous, stable and negative. In fact we have in this case:

$$\begin{aligned} &\text{improper automorphs:} && M = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ &\text{proper negautomorphs:} && N = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ &\text{improper negautomorphs:} && A = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{6.6}$$

As in table (4.3) one may distinguish between different types of primitive isotropic lattices. Note, however, that type IV lattice (enantiomorphic, unstable and negative) does not occur, because if $b = 2$, then c is odd and b divides $c^2 \pm 1$; the same is trivially true for $b = 1$. The general situation is illustrated in the following table.

TABLE (6.7)

Classification of isotropic primitive lattices with metric tensor $g(B) = (0, b, c)$

Lattice type	Characterization of the lattice Λ		Conditions on the metric tensor	Examples of $g(B)$
I	ambiguous stable negative	$\Lambda = \Lambda'$ $\Lambda = \bar{\Lambda}$ $\Lambda = \bar{\Lambda}'$	$b = 1$ or $b = 2$ and $c \equiv 1 \pmod 2$	$(0, 1, 0)$ and $(0, 2, -1) \overset{a}{\simeq} (1, 0, -1)$
II	ambiguous unstable positive	$\Lambda = \Lambda'$ $\Lambda \neq \bar{\Lambda}$ $\Lambda \neq \bar{\Lambda}'$	$b \mid c^2 - 1$ $b \nmid c^2 + 1$ $b \neq 1,2$	$(0, 3, -2)$
III	enantiomorphic stable positive	$\Lambda \neq \Lambda'$ $\Lambda = \bar{\Lambda}$ $\Lambda \neq \bar{\Lambda}'$	$b \nmid c^2 - 1$ $b \mid c^2 + 1$ $b \neq 1, 2$	$(0, 5, -2)$
V	enantiomorphic unstable positive	$\Lambda \neq \Lambda'$ $\Lambda \neq \bar{\Lambda}$ $\Lambda \neq \bar{\Lambda}'$	$b \nmid c^2 - 1$ $b \nmid c^2 + 1$ $b \neq 1, 2$	$(0, 7, -2)$

The conditions for an ambiguous primitive isotropic lattice to be rhombic or rectangular are expressed by the following proposition.

Proposition 16. An ambiguous primitive isotropic lattice with metric tensor $g(B) = (0, b, c)$ is rectangular if b is even and rhombic if b is odd.

Proof. Let us first remark that an ambiguous isotropic lattice is either rhombic or rectangular, because it admits only two mirrors $(\pm M)$, which then belong to the same arithmetic class.

Suppose the lattice rectangular; then:

$$(0, b, c) \overset{a}{\sim} (A, 0, C) \qquad \text{by} \qquad S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, Z)$$

giving the conditions:

$$2A\delta = \gamma b, \qquad -2C\gamma = \delta b, \qquad 2(\gamma c + \beta A) = -\alpha b,$$
$$2(\delta c - \alpha C) = -\beta b, \tag{6.8}$$

which implies

$$b \equiv 0 \pmod 2).$$

Suppose the lattice rhombic, then by S as above $(0, b, c) \overset{a}{\sim} (A, A, C)$, giving

$$\alpha b = A(2\delta - \gamma), \qquad \delta b = A(\delta - 2\gamma); \tag{6.9}$$

however, δ or γ is odd, thus:

$$b \equiv 1 \pmod 2).$$

7. *Bravais classes.* One has to distinguish between Bravais classes and relativistic Bravais classes. We have already defined the first concept by condition (2.10). The second arises only if one considers the identification of isomorphic space-time groups as discussed in ref. 3. In this section we consider only the subdivision of relativistic lattices into usual Bravais classes. Having these, it is very easy to go over to relativistic Bravais classes.

A first and important distinction can be made between relativistic lattices with holohedry of finite order (the ametric case) and lattices with holohedry infinite order (the metric case).

(a) Lattices with holohedries of finite order. In section 2 and 6 we have seen that lattices with metric tensors proportional to the integral ones, with discriminant not a squared integer, always have an infinite holohedry. Therefore only real incommensurable (a, b, c) and/or isotropic $(0, b, c)$ give rise to lattices with finite holohedries.

Proposition 17. The holohedry of a lattice is of finite order if and only if the lattice is isotropic and/or has an incommensurable metric tensor

$g(B) = (a, b, c)$. This means that for no real positiv k are ka, kb and kc rational integers.

Any lattice is left invariant by the identity E and by the total inversion $-E$. Lattices that are invariant only with respect to these two elements have holohedries that belong to a single arithmetic class, which defines the oblique Bravais class. The corresponding holohedry is $H_0 = \{-E\} \cong C_2$.

In the relativistic case, mirrors M are the only other lattice symmetries of finite order. We therefore consider ambiguous lattices. The holohedry of isotropic and/or incommensurable ambiguous lattices is:

$$H = \{-E, M\} \cong D_2. \tag{7.1}$$

In fact, two mirrors M_1 and M_2, where $M_1 \neq \pm M_2$ generate a proper Lorentz transformation of infinite order (cf. 3.4).

We have now to find out the arithmetic classes of (7.1). These are the same as the arithmetic classes of crystallographic Lorentz mirrors.

Proposition 18. There are two arithmetic classes of crystallographic improper Lorentz transformations: the rectangular class $\{M_R\}$ and the rhombic class $\{M_D\}$ where

$$M_R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_D = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Proof. Any Lorentz mirror is of the form:

$$M_\pm = \pm \begin{pmatrix} \cosh \phi & \sinh \phi \\ -\sinh \phi & -\cosh \phi \end{pmatrix}.$$

M_+ leaves invariant the space-like direction

$$\mu = k(\cosh \tfrac{1}{2}\phi\, e_1 - \sinh \tfrac{1}{2}\phi\, e_2)$$

with k real positive; and M_- leaves invariant the time-like direction

$$\rho = k(\sinh \tfrac{1}{2}\phi\, e_1 - \cosh \tfrac{1}{2}\phi\, e_2),$$

with k real positive, where as usual $g(B) = (1, 0, -1)$. One has

$$M_\pm \mu = \pm \mu \qquad \text{and} \qquad M_\pm \rho = \mp \rho. \tag{7.2}$$

$M_\pm$ being crystallographic, there is a $S \in GL(2, R)$ and a basis

$$BS = B' = (e_1', e_2'), \text{ such that:}$$

$$M_\pm' = S^{-1} M_\pm S \in GL(2, Z) \tag{7.3}$$

and there are lattice vectors in the μ and ρ directions (for more details, see refs. 21 and 22). According to the results of section 2 of this paper, there is, in the relativistic geometric class of lattices having the same $g(B')$, a lattice Λ left invariant by $M_\pm'$ which lattice has μ as first basis vector ($\mu = e_1''$). This means that there is a basis $B'' = (e_1'', e_2'' =)\ B'S'$ of Λ such that:

$$S'^{-1}M'_+S' = M''_+ = \begin{pmatrix} 1 & \gamma \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad \gamma \in Z. \tag{7.4}$$

The same can be said for M_- taking $e_1'' = \rho$. By:

$$\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \gamma \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \gamma + 2n \\ 0 & -1 \end{pmatrix}, \quad \forall n \in Z \tag{7.5}$$

one sees that:

$$\begin{pmatrix} 1 & \gamma \\ 0 & -1 \end{pmatrix} \overset{a}{\sim} \begin{pmatrix} 1 & \gamma \pmod 2 \\ 0 & -1 \end{pmatrix}. \tag{7.6}$$

Furthermore one verifies that M_R is not in the same arithmetic class as M_D. Finally

$$M_R \overset{a}{\sim} -M_R \quad \text{and} \quad M_D \overset{a}{\sim} -M_D. \tag{7.7}$$

To prove this, it is sufficient to interchange the rôle of ρ and μ in the demonstration above.

According to proposition 18, holohedries (7.1) belong to two arithmetic classes. Lattices having holohedries equivalent to the rectangular class belong to the rectangular Bravais class; those with holohedries equivalent to the rhombic class belong to the rhombic Bravais class. Note that invariance with respect to M_R implies $g(B) = (a, 0, c)$; invariance with respect to M_D gives $g(B) = (a, a, c)$. This explains the name of the class.

To conclude, there are the following three Bravais classes of lattices in the ametric and isotropic cases.

(i) The rectangular Bravais class. The lattices Λ_R belonging to this class have either an incommensurable or an isotropic metric tensor $g(B') = (a', b', c')$ such that there are real a, c with:

$$(a', b', c') \overset{a}{\sim} (a, 0, c), \qquad a > 0. \tag{7.8}$$

Their holohedry (with respect to a rectangular basis) is:

$$H_R = \{-E, M_R\} \cong D_2. \tag{7.9}$$

(ii) The rhombic class. The lattices Λ_D belonging to this class have either an incommensurable or an isotropic metric tensor $g(B') = (a', b', c')$ such that there are real a, c with

$$(a', b', c') \overset{a}{\sim} (a, a, c), \qquad a > 0. \tag{7.10}$$

Their holohedry (with respect to a rhombic basis) is:

$$H_D = \{-E, M_D\} \cong D_2. \tag{7.11}$$

(iii) The oblique Bravais class. All incommensurable or isotropic lattices that are neither in the rectangular nor in the rhombic Bravais class belong to the oblique Bravais class. Their holohedry is:

$$H_0 = \{-E\} \cong C_2. \tag{7.12}$$

(b) Lattices with holohedries of infinite order. As we have already seen in section 2, we may restrict our attention to primitive lattices whose discriminant is not a squared integer. In what follows by primitive lattices we mean non-isotropic primitive lattices.

According to proposition 3 any proper automorph of a primitive lattice is given by (3.14) and is therefore generated by the total inversion $-E$ and the fundamental Lorentz transformation L_0, which leave the lattice invariant.

The holohedry of an enantiomorphic primitive lattice is thus given by:

$$H_e = \{-E, L_0\} \cong C_\infty \times C_2. \tag{7.13}$$

Any improper automorph M of an ambiguous primitive lattice is the product of a particular mirror M_0 leaving the lattice invariant and a certain proper automorph. It follows that the holohedry of a primitive ambiguous lattice is:

$$H_a = \{-E, L_0, M_0\} \cong D_\infty \times C_2. \tag{7.14}$$

We do not have to investigate the arithmetic classes of (7.13) and of (7.14), for we know already the result:

The proper arithmetic classes of primitive indefinite quadratic forms are in one-to-one correspondence with the Bravais classes of metric lattices. In each Bravais class we have singled out by (2.13) a primitive lattice.

A list of chains of equivalent reduced quadratic forms corresponding to primitive lattices is given in the appendix for a number of discriminants. The table needs some comments.

Usually reduced quadratic forms are tabulated according to the increasing value of the discriminant. It is well known that the number of different arithmetic classes for a given discriminant is finite (see ref. 17, theorem 79, p. 103). The least positive solutions of Pell's equation are determined by a given discriminant.

In our classification, however, we have adopted another ordering criterion that is based on the kinematical interpretation of the proper Lorentz transformations leaving a relativistic lattice invariant. To a proper Lorentz transformation of infinite order there corresponds the transformation from an inertial frame to one moving with relative velocity v in the x direction (the only spatial dimension in two-dimensional Minkowskian space). The absolute value of v is determined by the trace $n \in Z$ of the Lorentz transformation considered. One has:

$$n = 2\cosh\phi = \frac{2}{\sqrt{1 - \left(\frac{v_n}{c}\right)^2}} \tag{7.15}$$

giving:

$$\boxed{v_n = \frac{\sqrt{n^2 - 4}}{n}\, c}\,. \tag{7.16}$$

We indicate below the correspondence for low values of n.

TABLE (7.17)

Kinematical interpretation of n						
n	3	4	5	6	7	8
$\frac{v_n}{c}$	$\frac{\sqrt{5}}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{21}}{5}$	$\frac{2\sqrt{2}}{3}$	$\frac{3\sqrt{5}}{7}$	$\frac{\sqrt{15}}{4}$
	0.745	0.866	0.916	0.942	0.958	0.968

From the physical point of view we expect that the lowest values of n are the most interesting ones. Therefore we have ordered the Bravais classes of primitive lattice according to increasing value of n.

For a given n it is easy (by looking at the factor decomposition of $n^2 - 4$) to determine all possible integral values of u and d in the Pell equation. Knowing the discriminant d, the corresponding arithmetic classes are derived by standard methods (see *e.g.* ref. 17).

8. *Concluding remarks.* By means of this paper we have laid down the foundation of a two-dimensional relativistic crystallography. The next step consists in determining of the arithmetic classes of relativistic point groups. These can then be used for the derivation of all two-dimensional space-time groups.

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APPENDIX

Bravais classes of primitive (non-isotropic) relativistic lattice (for $n \leq 25$)

n	u	d	Chains of equivalent primitive reduced forms	Type	Lattices
3	1	5	[1, 1, −1][−1, 1, 1]	ID	M_3 N_1
4	1	12	[1, 2, −2][−2, 2, 1] [2, 2, −1][−1, 2, 2]	IIRD	M_4 $\bar{M}_4$
5	1	21	[1, 3, −3][−3, 3, 1] [3, 3, −1][−1, 3, 3]	IID	M_5 $\bar{M}_5$
6	1	32	[1, 4, −4][−4, 4, 1] [4, 4, −1][−1, 4, 4]	IIRD	M_6 $\bar{M}_6$
	2	8	[1, 2, −1][−1, 2, 1]	IR	N_2
7	1	45	[1, 5, −5][−5, 5, 1] [5, 5, −1][−1, 5, 5]	IID	M_7 $\bar{M}_7$
	3	5	[1, 1, −1][−1, 1, 1]	ID	N_1
8	1	60	[1, 6, −6][−6, 6, 1] [6, 6, −1][−1, 6, 6]	IIRD	M_8 $\bar{M}_8$
			[2, 6, −3][−3, 6, 2] [3, 6, −2][−2, 6, 3]	IIRD	
9	1	77	[1, 7, −7][−7, 7, 1] [7, 7, −1][−1, 7, 7]	IID	M_9 $\bar{M}_9$
10	1	96	[1, 8, −8][−8, 8, 1] [8, 8, −1][−1, 8, 8]	IIRD	M_{10} $\bar{M}_{10}$
			[3, 6, −5][−5, 4, 4][4, 4, −5][−5, 6, 3] [5, 6, −3][−3, 6, 5][5, 4, −4][−4, 4, 5]	IIRD	
	2	24	[1, 4, −2][−2, 4, 1] [2, 4, −1][−1, 4, 2]	IIR	P_{24} $\bar{P}_{24}$
11	1	117	[1, 9, −9][−9, 9, 1] [9, 9, −1][−1, 9, 9]	IID	M_{11} $\bar{M}_{11}$
	3	13	[1, 3, −1][−1, 3, 1]	ID	N_3
12	1	140	[1, 10, −10][−10, 10, 1] [10, 10, −1][−1, 10, 10]	IIRD	M_{12} $\bar{M}_{12}$
			[2, 10, −5][−5, 10, 2] [5, 10, −2][−5, 10, 2]	IIRD	
13	1	165	[1, 11, −11][−11, 11, 1] [11, 11, −1][−1, 11, 11]	IID	M_{13} $\bar{M}_{13}$
			[3, 9, −7][−7, 5, 5][5, 5, −7][−7, 9, 3] [7, 9, −3][−3, 9, 7][7, 5, −5][−5, 9, −3]	IID	
14	1	192	[1, 12, −12][−12, 12, 1] [−1, 12, 12][12, 12, −1]	IIRD	M_{14} $\bar{M}_{14}$
			[3, 12, −4][−4, 12, 3] [−3, 12, 4][4, 12, −3]	IIRD	

(*Appendix continued*)

n	u	d	Chains of equivalent primitive reduced forms	Type	Lattices
14	2	48	[1, 6, −3][−3, 6, 1] [−1, 6, 3][3, 6, −1]	IIR	P_{48} $\bar{P}_{48}$
	4	12	[1, 2, −2][−2, 2, 1] [−1, 2, 2][2, 2, −1]	IIRD	M_4 $\bar{M}_4$
15	1	221	[1, 13, −13][−13, 13, 1] [−1, 13, 13][13, 13, −1]	IID	M_{15} $\bar{M}_{15}$
			[5, 11, −5][−5, 9, 7][7, 5, −7][−7, 9, 5] [−5, 11, 5][5, 9, −7][−7, 5, 7][7, 9, −5]	III	
16	1	252	[1, 14, −14][−14, 14, 1] [−1, 14, 14][14, 14, −1]	IIRD	M_{16} $\bar{M}_{16}$
			[2, 14, −7][−7, 14, 2] [−2, 14, 7][7, 14, −2]	IIRD	
	3	28	[1, 4, −3][−3, 2, 2][2, 2, −3][−3, 4, 1] [−1, 4, 3][3, 2, −2][−2, 2, 3][3, 4, −1]	IIRD	P_{28} $\bar{P}_{28}$
17	1	285	[1, 15, −15][−15, 15, 1] [−1, 15, 15][15, 15, −1]	IID	M_{17} $\bar{M}_{17}$
			[3, 15, −5][−5, 15, 3] [−3, 15, 5][5, 15, −3]	IID	
18	1	320	[1, 16, −16][−16, 16, 1] [−1, 16, 16][16, 16, −1]	IIRD	M_{18} $\bar{M}_{18}$
			[4, 12, −11][−11, 10, 5][5, 10, −11][−11, 12, 4] [−4, 12, 11][11, 10, −5][−5, 10, 11][11, 12, −4]	IIRD	
18	2	80	[1, 8, −4][−4, 8, 1] [−1, 8, 4][4, 8, −1]	IIR	P_{80} $\bar{P}_{80}$
	4	20	[1, 4, −1][−1, 4, 1]	IR	N_4
	8	5	[1, 1, −1][−1, 1, 1]	ID	N_1
19	1	357	[1, 17, −17][−17, 17, 1] [−1, 17, 17][17, 17, −1]	IID	M_{19} $\bar{M}_{19}$
			[3, 15, −11][−11, 7, 7][7, 7, −11][−11, 15, 3] [−3, 15, 11][11, 7, −7][−7, 7, 11][11, 15, −3]	IID	
20	1	396	[1, 18, −18][−18, 18, 1] [−1, 18, 18][18, 18, −1]	IIRD	M_{20} $\bar{M}_{20}$
			[2, 18, −9][−9, 18, 2] [−2, 18, 9][9, 18, −2]	IIRD	
			[5, 16, −7][−7, 12, 9][9, 6, −10][−10, 14, 5] [−5, 16, 7][7, 12, −9][−9, 6, 10][10, 14, −5] [−7, 16, 5][5, 14, −10][−10, 6, 9][9, 12, −7] [7, 16, −5][−5, 14, 10][10, 6, −9][−9, 12, 7]	V	
	3	44	[1, 6, −2][−2, 6, 1] [−1, 6, 2][2, 6, −1]	IIRD	P_{44} $\bar{P}_{44}$

(*Appendix continued*)

n	u	d	Chains of equivalent primitive reduced forms	Type	Lattices
21	1	437	[1, 19, −19][−19, 19, 1] [−1, 19, 19][19, 19, −1]	IID	M_{21} $\overline{M}_{21}$
22	1	480	[1, 20, −20][−20, 20, 1] [−1, 20, 20][20, 20, −1]	IIRD	M_{22} $\overline{M}_{22}$
			[4, 20, −5][−5, 20, 4] [−4, 20, 5][5, 20, −4]	IIRD	
			[8, 16, −7][−7, 12, 12][12, 12, −7][−7, 16, 8] [−8, 16, 7][7, 12, −12][−12, 12, 7][7, 16, −8]	IIRD	
			[3, 18, −13][−13, 8, 8][8, 8, −13][−13, 18, 3] [−3, 18, 13][13, 8, −8][−8, 8, 13][13, 8, −3]	IIRD	
	2	120	[1, 10, −5][−5, 10, 1] [−1, 10, 5][5, 10, −1]	IIR	P_{120} $\overline{P}_{120}$
			[2, 8, −7][−7, 6, 3][3, 6, −7][−7, 8, 2] [−2, 8, 7][7, 6, −3][−3, 6, 7][7, 8, −2]	IIR	
23	1	525	[1, 21, −21][−21, 21, 1] [−1, 21, 21][21, 21, −1]	IID	M_{23} $\overline{M}_{23}$
			[3, 21, −7][−7, 21, 3] [−3, 21, 7][7, 21, −3]	IID	
	5	21	[1, 3, −3][−3, 3, 1] [−1, 3, 3][3, 3, −1]	IID	P_{21} $\overline{P}_{21}$
24	1	572	[1, 22, −22][−22, 22, 1] [−1, 22, 22][22, 22, −1]	IIRD	M_{24} $\overline{M}_{24}$
			[2, 22, −11][−11, 22, 2] [−2, 22, 11][11, 22, −2]	IIRD	
25	1	621	[1, 23, −23][−23, 23, 1] [−1, 23, 23][23, 23, −1]	IID	M_{25} $\overline{M}_{25}$
			[5, 21, −9][−9, 15, 11][11, 7, −13][−13, 19, 5] [−5, 21, 9][9, 15, −11][−11, 7, 13][13, 19, 5] [−9, 21, 5][5, 19, −13][−13, 7, 11][11, 15, −9] [9, 21, −5][−5, 19, 13][13, 7, −11][−11, 15, 9]	V	
	3	69	[1, 7, −5][−5, 3, 3][3, 3, −5][−5, 7, 1] [−1, 7, 5][5, 3, −3][−3, 3, 5][5, 7, −1]	IID	P_{69} $\overline{P}_{69}$

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