Physica 41 (1969) 541–565

CRYSTALLOGRAPHIC GROUPS IN SPACE AND TIME

I. GENERAL DEFINITIONS AND BASIC PROPERTIES

T. JANSSEN and A. JANNER

Instituut voor Theoretische Fysika, Katholieke Universiteit, Nijmegen, Nederland

and

E. ASCHER

Battelle Institute, Advanced Studies Center, Genève, Suisse

Received 4 June 1968

Synopsis

A generalization of the concept of n-dimensional magnetic group is considered which also admits discrete time translations. This leads to the study of crystallographic groups in $n+1$-dimensional Euclidean, Minkowskian, Galilean and product spaces. Definitions are given for point groups, system groups, arithmetic and geometric crystal classes, Bravais classes, lattice systems and space(-time) groups in these spaces. As in the Euclidean case, space-time groups G^{n+1} appear in (K, Z^{n+1}, ϕ)-extensions with K a crystallographic point group and ϕ a monomorphism ϕ: $K \to GL(n+1, Z)$. As it is not yet known under which conditions groups appearing in such extensions may be interpreted as space-time groups, the classification of these groups is here restricted to the case of finite K. This classification arises by identifying space(-time) groups related by an isomorphism which takes into due account the various kinds of translation elements. For known geometric point groups a constructive method to derive all non-isomorphic space(-time) groups is given. The number of Bravais classes in Euclidean and Galilean space turns out to be finite. It is enumerably infinite in so-called product space and continuously infinite in Minkowskian space. The same is true for the number of non-isomorphic space(-time) groups.

1. *Introduction*

In recent years an extensive study has been made of magnetic groups[1,2]. These are n-dimensional crystallographic groups in which time inversion is considered next to and in combination with space transformations.

If one admits also discrete time translations one is led to the study of $n+1$-dimensional crystallographic groups. The problem is to find a suitable space on which these groups act as transformation groups. Consider the simplest possibility: a $n+1$-dimensional Euclidean vector space which is a vector space with a positive definite metric. The group of inhomogeneous real linear transformations leaving this metric invariant is the Euclidean group $E(n + 1)$. An $n + 1$-dimensional space group is a discrete subgroup of the Euclidean group which contains as a maximal abelian subgroup a translation group generating an $n + 1$-dimensional lattice. As Ascher and Janner[3]) have discussed, such a space group G^{n+1} may be obtained from an extension:

$$0 \to Z^{n+1} \to G^{n+1} \to K \to 1 \qquad (\phi) \tag{1.1}$$

of a free abelian group Z^{n+1} by a finite group K called point group. The operation of K on Z^{n+1} is given by a monomorphism $\phi: K \to GL(n + 1, Z)$. In crystallography it is customary to identify affine equivalent space groups. According to Bieberbach[4]) two space groups are affine equivalent if and only if they are isomorphic.

However, the Euclidean space does not take into account the difference between time axis and space axes. The solution of the problem would be considering the $n + 1$-dimensional Minkowskian vector space which has an indefinite metric of signature $(n, 1)$. The homogeneous real linear group leaving this metric invariant is the Lorentz group, the inhomogeneous one is the Poincaré group. A relativistic space-time group is a discrete subgroup of the Poincaré group with a maximal abelian subgroup of translations generating an $n + 1$-dimensional lattice. Relativistic space-time groups can also be obtained from extensions (1.1), but the general conditions under which groups appearing in extensions (1.1) may be interpreted as relativistic space-time groups are under investigation. Because in Minkowskian space one has to distinguish between spacelike, timelike and isotropic vectors, equivalence of two space-time groups is a stronger equivalence relation than plain isomorphism as groups. We call two such groups isomorphic if there exists an isomorphism between them which maps translations on translations of the same kind. To find all non-isomorphic relativistic space-time groups is, as far as we know, an unsolved problem even in the lowest dimension ($n = 1$). This is a consequence of the indefinite character of the metric.

In the non-relativistic limit the Minkowskian vector space is transformed into the Galilean vector space with a singular metric. In this same limit the Poincaré group becomes the inhomogeneous Galilean group. A Galilean space-time group is a discrete subgroup of the inhomogeneous Galilean group with a maximal abelian subgroup of translations generating an $n + 1$-dimensional lattice. In Galilean space one distinguishes space type vectors ($n + 1^{\text{th}}$ component zero) and velocity type vectors ($n + 1^{\text{th}}$ component different from zero). Two Galilean space-time groups are called isomorphic if there is a group isomorphism between them which maps translation elements on translations of the same kind.

Regarding the difficulties involved in the determination of relativistic and Galilean space-time groups it is worthwhile to consider a last possibility: the direct product of an n-dimensional Euclidean space (interpreted as space) and a one-dimensional Euclidean space (representing the time). We simply call it product space. Actually this is the type of vector space one considers implicitly when speaking of magnetic groups. The group of inhomogeneous linear transformations leaving invariant the metric in both spaces is called the inhomogeneous pseudo-Lorentz group. A generalized magnetic space-time group (GM space-time group) is a discrete subgroup of the inhomogeneous pseudo-Lorentz group $JP(n+1)$ with a maximal abelian subgroup of translations generating an $n+1$-dimensional lattice. In product space one may distinguish three kinds of vectors: those lying in one of the two spaces and a mixed type (with non-vanishing components in both spaces). Two GM space-time groups are called isomorphic if there exists a group isomorphism between them mapping translation elements of a given type on translation elements of the same type.

The present part presents a study of the properties and the classification of the space-(time) groups in the three first named different spaces. The classification of GM space-time groups is not further treated, but this may be done quite analogously. Some preliminary results are published in technical reports [5]).

Before starting the discussion of the various crystallographic groups a brief survey of the corresponding continuous transformation groups is given.

2. *Metric spaces and transformation groups*

Consider an $n+1$-dimensional real vector space R^{n+1}. Provided with a positive definite metric this is a *Euclidean space*. – If an orthonormal basis $e_1, e_2, \ldots, e_{n+1}$ is chosen a vector $x \in R^{n+1}$ has real components $(x^1, \ldots, x^{n+1})$ and its norm is given by the non-degenerate quadratic form

$$\|x\|^2 = \sum_{i=1}^{n+1} (x^i)^2.$$

The non-singular linear transformations leaving this metric invariant form the orthogonal group $O(n+1)$. The inhomogeneous transformations leaving the norm of the difference of any two vectors invariant form the group of solid motions, the Euclidean group $E(n+1)$. The uniquely determined maximal abelian normal subgroup of $E(n+1)$ is the $n+1$-dimensional translation group T^{n+1}. $E(n+1)$ is the semi-direct product

$$E(n+1) = T^{n+1} \wedge O(n+1)$$

where $O(n+1)$ acts on T^{n+1} in the natural way. Denoting the elements of $E(n+1)$ by (a, α), where $a \in T^{n+1}$ and $\alpha \in O(n+1)$, multiplication in $E(n+1)$ is defined by

$$(a, \alpha)(b, \beta) = (a + \alpha b, \alpha\beta)$$

for all $a, b \in T^{n+1}$ and all $\alpha, \beta \in O(n+1)$. By αb is meant $\phi(\alpha)\, b$, where ϕ is here the natural monomorphism $O(n+1) \to \mathrm{Aut}(T^{n+1})$.

Providing R^{n+1} with an indefinite metric of signature $(n, 1)$ one has an $n+1$-dimensional *Minkowskian space*. – Choosing an orthonormal basis $e_1, \ldots, e_{n+1}$ in such a way that

$$\|e_i\|^2 = 1 \qquad (i = 1, \ldots, n)$$
$$\|e_{n+1}\|^2 = -1$$

the norm of a vector with components $x^1, \ldots, x^{n+1}$ is given by

$$\|x\|^2 = \sum_{i=1}^{n} (x^i)^2 - (x^{n+1})^2.$$

The orthogonal group of this metric is the homogeneous Lorentz group $O(n, 1)$. The inhomogeneous Lorentz group or Poincaré group $JL(n+1)$ is the semi-direct product

$$JL(n+1) = T^{n+1} \hat{} \, O(n, 1)$$

where $O(n, 1)$ operates on T^{n+1} in the natural way.

By choosing another orthogonal basis in Minkowskian space by

$$\begin{aligned} e_i' &= e_i \qquad && (i = 1, \ldots, n) \\ e_{n+1}' &= c e_{n+1} \qquad && (\text{real positive } c) \end{aligned} \tag{2.1}$$

one obtains in the limit $c \to \infty$ the $n+1$-dimensional *Galilean space* with a singular metric. The corresponding limit of the Lorentz group is the Galilean group $G(n+1)$ which is obtained in the following way. By $O^+(n, 1)$ we denote the one-component of $O(n, 1)$ and by V:

– for odd n: the group generated by

$$\begin{pmatrix} -\mathbb{1}_n & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -1 \end{pmatrix}$$

– for even n: the group generated by

$$\begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} M_n & 0 \\ 0 & 1 \end{pmatrix} \text{ with } M_n = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_{n-1} \end{pmatrix}.$$

Then $O(n, 1)$ is generated by $O^+(n, 1)$ and V.

Any $\lambda \in O^+(n, 1)$ may be written as:

$$\lambda(\chi) = \begin{bmatrix} \mathrm{P}_n & & 0 \\ & & \vdots \\ & & 0 \\ 0\ldots0 & & 1 \end{bmatrix} \begin{bmatrix} \cosh\chi & 0 & \sinh\chi \\ 0 & \mathbb{1}_{n-1} & 0 \\ \sinh\chi & 0 & \cosh\chi \end{bmatrix} \begin{bmatrix} Q_n & & 0 \\ & & \vdots \\ & & 0 \\ 0\ldots0 & & 1 \end{bmatrix}$$

where P_n and Q_n are n-dimensional orthogonal transformations. After the basis transformation (2.1) and putting $|v| = c\chi$ one has:

$$\lim_{c\to\infty} \lambda'\left(\frac{v}{c}\right) = \begin{pmatrix} \mathrm{P}_n Q_n & v \\ 0 & 1 \end{pmatrix} \equiv \gamma(v)$$

where v is a n-dimensional column vector; $\gamma(v)$ is an element of $G(n+1)$, and $G(n+1)$ is generated by what we denote as $\lim_{c\to\infty} O^+(n, 1)$ and V.

The inhomogeneous Galilean group is the semi-direct product

$$JG(n+1) = T^{n+1} \wedge G(n+1)$$

where again $G(n+1)$ acts on T^{n+1} in the natural way.

We call *product space* the direct product of an n-dimensional and a one-dimensional Euclidean space. The metric in the first space is given by:

$$x^2 = \sum_{i=1}^{n} (x^i)^2$$

in the second one by

$$x^2 = (x^{n+1})^2.$$

The linear transformations leaving invariant the metric in both spaces form the homogeneous pseudo-Lorentz group $\underline{O}(n, 1)$.

Proposition 1

$$\underline{O}(n, 1) = O(n) \times O(1).$$

Proof: Choose an orthonormal basis with $e_1, \ldots, e_n$ in the first space and e_{n+1} in the second one. Then an element $T \in \underline{O}(n, 1)$ is a matrix such that for any x with components $x^1, \ldots, x^{n+1}$:

$$\sum_{i=1}^{n} (x^i)^2 = \sum_{j=1}^{n} \sum_{k,l=1}^{n+1} T_{jk} T_{jl} x^k x^l$$

and

$$(x^{n+1})^2 = \sum_{k,l=1}^{n+1} T_{n+1,k} T_{n+1,l} x^k x^l.$$

From this follows

$$\sum_{j=1}^{n} T_{jk} T_{jl} = \delta_{kl} \qquad (k, l \leqslant n),$$

$$T_{j,n+1} = T_{n+1,k} = 0 \qquad (j, k \leqslant n),$$

$$T^2_{n+1,n+1} = 1.$$

Therefore the matrix T has the form

$$T = \begin{pmatrix} \mathrm{P} & 0 \\ 0 & \pm 1 \end{pmatrix} \tag{2.2}$$

where P is an orthogonal matrix.

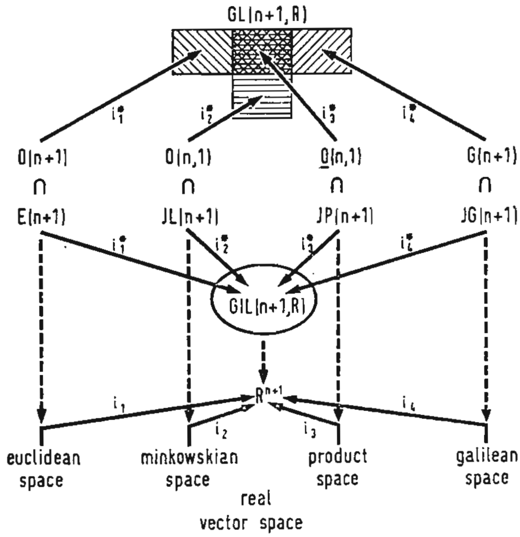


Fig. 1

Proposition 2

The pseudo-Lorentz group leaves invariant all quadratic forms

$$\alpha \sum_{i=1}^{n} (x^i)^2 + \beta(x^{n+1})^2 \qquad \text{(all real } \alpha \text{ and } \beta).$$

This is a direct consequence of proposition 1. In particular $\underline{O}(n, 1)$ leaves invariant the quadratic forms

$$\sum_{i=1}^{n+1} (x^i)^2 \quad \text{and} \quad \sum_{i=1}^{n} (x^i)^2 - (x^{n+1})^2.$$

The inhomogeneous pseudo-Lorentz group $JP(n + 1)$ is the semi-direct product:

$$JP(n + 1) = T^{n+1} \wedge \underline{O}(n, 1).$$

By choosing a basis in each of the spaces discussed one obtains isomorphic mappings onto R^{n+1} considered as real vector space. These mappings induce isomorphic mappings of the groups $E(n + 1)$, $JL(n + 1)$, $JP(n + 1)$ and $JG(n + 1)$ into the real affine group $GIL(n + 1, R)$, which is the semi-direct product of R^{n+1} by the real general linear group $GL(n + 1, R)$.

For the rest of the paper we suppose that a fixed orthonormal basis is chosen, if not stated otherwise. As $n + 1^{\text{th}}$ axis we take:
- in Minkowskian space the direction of the basis vector of norm -1;
- in Galilean space the direction obtained in its limit (2.1);
- in product space the one-dimensional Euclidean space.

The $n + 1^{\text{th}}$ axis is sometimes called t axis.

Once these bases are chosen the transformation groups may be identified with their isomorphic images in $GIL(n+1, R)$.

Proposition 3:

$\underline{O}(n, 1) = O(n+1) \cap O(n, 1) = O(n+1) \cap G(n+1) = O(n, 1) \cap G(n+1)$.
The proof is a direct consequence of proposition 1 and the fact that the elements of $G(n+1)$ have the matrix form

$$G = \begin{bmatrix} R & \cdot & v^1 \\ & & \vdots \\ & & v^n \\ 0 \ldots 0 & & \pm 1 \end{bmatrix} \tag{2.3}$$

where $R \in O(n)$.

Because in each space the group of solid motions is the semi-direct product of the translation group by the orthogonal group of the space one has:

Proposition 4:

$$\begin{aligned} JP(n+1) &= E(n+1) \cap JL(n+1) = \\ &= E(n+1) \cap JG(n+1) = \\ &= JL(n+1) \cap JG(n+1). \end{aligned}$$

3. *Euclidean space groups*

We give here a brief summary of those properties of Euclidean space groups which form a fundament for the subsequent generalization. At the same time we introduce some new concepts. We refer to refs. 3 to 5 for more details and for the proofs.

An $n+1$-dimensional *space group* G^{n+1} is a subgroup of the Euclidean group $E(n+1)$ with $U^{n+1} \stackrel{\text{def}}{=} T^{n+1} \cap G^{n+1}$ a free abelian subgroup of rank $n+1$ which generates over R the vector space T^{n+1} (*i.e.* $RU^{n+1} = T^{n+1}$). Let σ be the restriction to G^{n+1} of the epimorphism $\sigma' \colon E(n+1) \twoheadrightarrow O(n+1)$ and put $K = \operatorname{Im} \sigma$. Then it follows from the definition that U^{n+1} is normal in G^{n+1}, that $G^{n+1}/U^{n+1} \cong K \subset O(n+1)$ and that K acts effectively on U^{n+1}. Furthermore U^{n+1} is maximal abelian and of finite index in G^{n+1}. A set of points equivalent by the operation of U^{n+1} is called a $n+1$-dimensional *lattice* Λ (*i.e.* $U^{n+1}x_0 = \Lambda$). K is a *point group* leaving Λ invariant.

Each space group appears in an extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^{n+1} & \stackrel{\kappa}{\rightarrowtail} & G^{n+1} & \stackrel{\sigma}{\twoheadrightarrow} & K & \longrightarrow & 1 \qquad (\phi)^{*} \\ & & \lambda \downarrow & \nearrow i & & & & & \\ & & U^{n+1} & & & & & & \end{array} \tag{3.1}$$

* In diagrams we often use the following notation: a homomorphism is denoted by $\longrightarrow$, a monomorphism by $\rightarrowtail$, an epimorphism by $\twoheadrightarrow$, an isomorphism by $\rightarrowtail\!\!\!\!\twoheadrightarrow$.

of Z^{n+1} by K with $\phi: K \to GL(n+1, Z)$ a monomorphism. The subgroup U^{n+1} is unique, so the monomorphism $\kappa = i \circ \lambda$ is only variable in the isomorphism λ. Fixing λ is equivalent to choosing a basis of Λ or correspondingly a set of free generators of U^{n+1}. The monomorphism ϕ gives a $n+1$-dimensional integral faithful representation of K. The group $\phi(K)$ is called an arithmetic point group.

Consider next to the diagram (3.1) the commutative diagram

$$\begin{array}{ccc} Z^{n+1} & \xrightarrow{\lambda} & \\ \downarrow \chi & & U^{n+1} \\ Z^{n+1} & \xrightarrow{\bar{\lambda}} & \end{array} \tag{3.2}$$

where λ and $\bar{\lambda}$ are isomorphisms, χ an automorphism of Z^{n+1}, *i.e.*

$$\chi \in GL(n+1, Z).$$

$\bar{\lambda}$ corresponds to another choice of basis U^{n+1}. Then (3.2) induces a morphism of group extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^{n+1} & \xrightarrow{\kappa} & G^{n+1} & \xrightarrow{\sigma} & K & \longrightarrow & 1 \qquad (\phi) \\ & & \chi \downarrow & & \| & & \| & & \\ 0 & \longrightarrow & Z^{n+1} & \xrightarrow{\bar{\kappa}} & G^{n+1} & \xrightarrow{\sigma} & K & \longrightarrow & 1 \qquad (\bar{\phi}) \end{array} \tag{3.3}$$

such that

$$\bar{\phi}(\alpha) = \chi\phi(\alpha)\,\chi^{-1}. \qquad (\forall \alpha \in K)$$

This means that ϕ and $\bar{\phi}$ are two Z-equivalent representations of K. The condition of arithmetic equivalence for two point groups $\phi(K)$ and $\bar{\phi}(K)$ is weaker:

$$\bar{\phi}(K) = \chi\phi(K)\,\chi^{-1} \tag{3.4}$$

and arises from the consideration of isomorphic space groups.

According to ref. 3, proposition 6, given two isomorphic space groups G^{n+1} and $\bar{G}^{n+1}$, there exists a morphism of group extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^{n+1} & \xrightarrow{\kappa} & G^{n+1} & \xrightarrow{\sigma} & K & \longrightarrow & 1 \qquad (\phi) \\ & & \chi \downarrow & & \psi \downarrow & & \omega \downarrow & & \\ 0 & \longrightarrow & Z^{n+1} & \xrightarrow{\bar{\kappa}} & \bar{G}^{n+1} & \xrightarrow{\bar{\sigma}} & K & \longrightarrow & 1 \qquad (\phi) \end{array} \tag{3.5}$$

such that

$$\bar{\phi}(\omega\alpha) = \chi\phi(\alpha)\,\chi^{-1} \qquad (\forall \alpha \in K)$$

with $\chi \in GL(n+1, Z)$. This means that isomorphic space groups determine the same *arithmetic class*, *i.e.* a class of arithmetically equivalent point groups. As shown in ref. 3 in the lower extension one may always choose $\phi(K) = \bar{\phi}(\bar{K})$. We call $\phi(K)$ an arithmetic point group. Then denoting by

$N_{\phi(K)}$ the normalizer of $\phi(K)$ in $GL(n+1, Z)$:

$$\chi \in N_{\phi(K)} \stackrel{\text{def}}{=\!=} \{\chi \in GL(n+1, Z) | \chi\phi(K)\, \chi^{-1} = \phi(K)\},$$

and one has the morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^{n+1} & \xrightarrow{\kappa} & G^{n+1} & \xrightarrow{\sigma} & K & \longrightarrow & 1 \qquad (\phi) \\ & & \chi \downarrow & & \psi \downarrow & & \omega \downarrow & & \\ 0 & \longrightarrow & Z^{n+1} & \xrightarrow{\bar{\kappa}} & \bar{G}^{n+1} & \xrightarrow{\bar{\sigma}} & \bar{K} & \longrightarrow & 1 \qquad (\bar{\phi}) \end{array}$$

Hence one obtains all non-isomorphic space groups by finding all non-equivalent extensions of Z^{n+1} by one representative of each arithmetic crystal class.

The first problem is to find the arithmetic crystal classes. An element of a space group may be written as (a, α) with $a \in T^{n+1}$ and $\alpha \in O(n+1)$. For a given space group the elements α form the corresponding point group K. In $GL(n+1, R)$ the point groups of two affine equivalent spaces are conjugate by a non-singular matrix and therefore similar. But two similar point groups are even conjugate by an orthogonal transformation (ref. 6, p. 47). Therefore similar point groups are conjugate subgroups of $O(n+1)$, *i.e.* geometrically equivalent. The equivalence class is called geometric crystal class.

For a given point group K one defines a subset L_K of the set L of $n+1$-dimensional lattices Λ as follows:

$$L_K = \{\Lambda \in L | K\Lambda = \Lambda\}.$$

Because of eqs. (3.1) and (3.4) a pair (K, Λ) with $\Lambda \in L_K$ defines an arithmetic crystal class.

Proposition 5:

If the point groups K and $\bar{K}$ are geometrically equivalent, then for each $\Lambda \in L_K$ there is a $\bar{\Lambda} \in L_{\bar{K}}$ such that (K, Λ) and $(\bar{K}, \bar{\Lambda})$ determine the same arithmetic crystal class.

Proof:

$$\bar{K} = TKT^{-1} \qquad \text{for some} \qquad T \in O(n+1),$$

$$\bar{K}(T\Lambda) = T\Lambda, \quad \text{so} \quad T\Lambda \in L_{\bar{K}}.$$

On bases $B(\Lambda)$ of Λ and $TB(\Lambda)$ of $T\Lambda$ both K and $\bar{K}$ correspond to the same group of matrices $\phi(K)$.

Consequence

To obtain one representative from each arithmetic crystal class it is sufficient to consider one representative from each geometric class and to determine all arithmetic representations on bases of lattices left invariant by this representative.

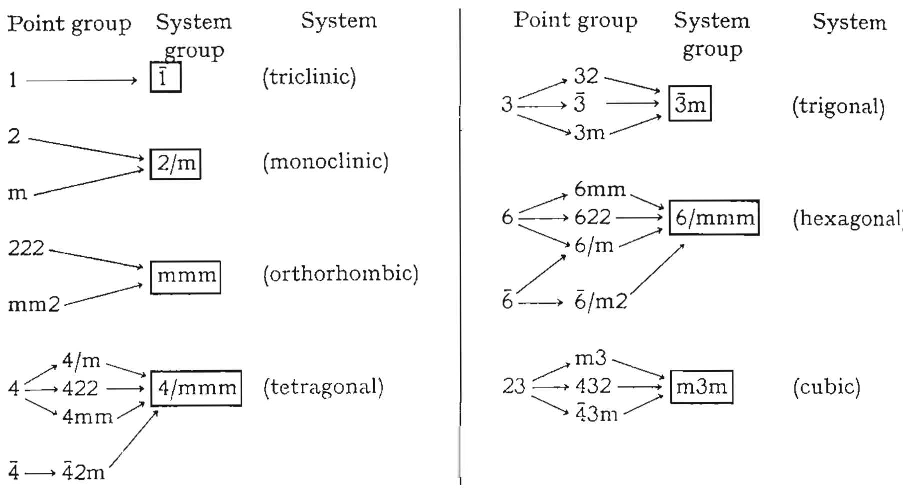
The maximal point group leaving a lattice invariant is its *holohedry* H. A lattice Λ determines an arithmetic crystal class with representative $\phi(H)$, but the converse is not true. The arithmetic crystal class of the holohedry defines the *Bravais class* of Λ. Two lattices belong to the same Bravais class if and only if their arithmetic holohedries are arithmetically equivalent. In the same way the geometric class of the holohedry defines the *system* of Λ. Two lattices belong to the same system if and only if their holohedries are geometrically equivalent.

For a given point group K a point group $\bar{K}$ is called enveloping if for every $\Lambda \in L_K$ one has a group from the geometric class of $\bar{K}$ which leaves Λ invariant. The maximal enveloping group of K is called its *system group* K_0. The reason of this terminology is that, for $n \leqslant 3$, each system group K_0 is the holohedry of at least one lattice from L_{K_0}. So in these cases the system groups and geometric holohedries are the same.

Example: 3-dimensional geometric groups, enveloping groups and system groups. Notation:

$A \to B$: B is an enveloping group for A.

$\boxed{C}$: C is a system group.



The set L_K is divided in two subsets: to the first subset belong those lattices for which K_0 is the holohedry, to the second one those for which this is not the case. In both subsets, $L^{\mathrm{I}}_{K_0}$ and $L^{\mathrm{II}}_{K_0}$, an equivalence relation is defined:

i) in $L^{\mathrm{I}}_{K_0}$ two lattices Λ_1 and Λ_2 are equivalent if (K_0, Λ_1) and (K_0, Λ_2) determine the same arithmetic crystal class; these classes correspond to Bravais classes.

ii) in $L^{II}_{K_0}$ Λ_1 and Λ_2 are equivalent, if they belong to the same Bravais class.

The different arithmetic crystal classes corresponding to K_0 are found as follows. From each class in $L^{I}_{K_0}$ one chooses one representative Λ and considers (K_0, Λ); among the classes of $L^{II}_{K_0}$ one considers only those for which the holohedry $\phi(H_1)$ has not a subgroup arithmetically equivalent to the holohedry $\phi(H_2)$ of another class. From each of these classes one chooses one representative Λ and considers (H, Λ). The different arithmetic point groups $\phi(K_0)$ are arithmetically equivalent to a subgroup of a group from the class (H, Λ).

Example: The 3-dimensional system group $K_0 = \bar{3}m$ leaves invariant lattices of the trigonal, hexagonal and cubic systems; $\bar{3}$m is holohedry for the rhombohedral lattices of the trigonal system. If we denote the arithmetic classes by the corresponding symmorphic space group symbol, this defines the arithmetic point group $R\,\bar{3}$m, which is an arithmetic holohedry. The arithmetic holohedries of the primitive, body-centered and face-centered cubic lattices have a subgroup arithmetically equivalent to $R\,\bar{3}$m. So the arithmetic point groups corresponding to $\bar{3}$m and not arithmetically equivalent to $R\,\bar{3}$m are the arithmetically non-equivalent subgroups of an arithmetic holohedry of a hexagonal lattice. There are two of them, $P\,\bar{3}$m1 and $P\,\bar{3}$m1. So all together there are three arithmetic crystal classes corresponding to $\bar{3}$m.

Proposition 6:

If K_0 is the system group of K, then each arithmetic group $\phi(K)$ is arithmetically equivalent to a subgroup of $\phi(K_0)$.

Proof: $\phi(K)$ is an integral faithful representation of K related to a basis of a lattice Λ, such that $K\Lambda = \Lambda$. Then also $K_0\Lambda = \Lambda$. As $K \subseteq K_0$, also $\phi(K) \subseteq \phi(K_0)$.

Note that $\phi(K_0)$ may contain several arithmetic point groups $\phi(K)$ which are arithmetically non-equivalent. As a consequence of proposition 6 it is sufficient to find the equivalence classes in L_{K_0} for each system group K_0 in order to obtain all arithmetically non-equivalent $\phi(K_0)$. Afterwards one determines for each subgroup $K \subset K_0$ arithmetically non-equivalent subgroups $\phi(K)$ of $\phi(K_0)$.

Finally one has to find all non-isomorphic extensions (3.1) for a given arithmetic point group $\phi(K)$. (For the theory of group extensions see refs. 7–10.) Suppose the point group K to be given by generators $\alpha_1, \ldots, \alpha_r$ and defining relations

$$\phi_i(\alpha_1, \ldots, \alpha_r) = \varepsilon \qquad (i = 1, \ldots, r)$$

and suppose that $r(\alpha)$ is a representative of the coset of Z^{n+1} in G^{n+1} which is mapped by σ (3.1) on $\alpha \in K$. Then the words

$$\kappa g_i = \phi_i(r(\alpha_1), \ldots, r(\alpha_\nu)) \qquad (i = 1, \ldots, r)$$

belong to κZ^{n+1}, see ref. 6. In order to define a group G^{n+1} in the extension these g_i have to be solutions of

$$\sum_{i=1}^{r} h_i g_i = 0 \tag{3.6}$$

where $H = [h_1, \ldots, h_r]$ are called the extension conditions for the group K, see refs. 6 and 10. If $\{g_i\}_{i=1,\ldots,r}$ satisfy eq. (3.6), the space group G^{n+1} is generated by the free generators $\kappa a_1, \ldots, \kappa a_{n+1}$ of κZ^{n+1}, the representatives $r(\alpha_1), \ldots, r(\alpha_\nu)$ and the relations

$$\begin{cases} \kappa a_i + \kappa a_j = \kappa a_j + \kappa a_i & (i, j = 1, \ldots, n+1), \\ \phi_i(r(\alpha_1), \ldots, r(\alpha_\nu)) = \kappa g_i & (i = 1, \ldots r), \\ r(\alpha) + \kappa a - r(\alpha) = \kappa \alpha a \equiv \kappa \phi(\alpha) a & (a \in Z^{n+1}, \alpha \in K). \end{cases} \tag{3.7}$$

In the same way $\bar{G}^{n+1}$ is generated by $\bar{\kappa} a_1, \ldots, \bar{\kappa} a_{n+1}$ and $\bar{r}(\alpha_1), \ldots, \bar{r}(\alpha_\nu)$. If an isomorphism $\psi: G^{n+1} \to \bar{G}^{n+1}$ exists such that

$$\psi r(\alpha) = \bar{\kappa} c(\alpha) + \bar{r}(\alpha) \qquad (\forall \alpha \in K)$$

for certain $c \in C^1_\phi(K, Z^{n+1})$, the extensions are equivalent. The equivalence classes of (K, ϕ)-extensions of Z^{n+1} form an abelian group $\mathrm{Ext}(K, Z^{n+1}, \phi)$ (see ref. 9), which is isomorphic to the second cohomology group $H^2_\phi(K, Z^{n+1})$. Two groups G^{n+1} and $\bar{G}^{n+1}$ appearing in equivalent extensions (3.5) are isomorphic. However also two groups belonging to non-equivalent extensions may be isomorphic. In this case they may be obtained from non-equivalent extensions with the same $\phi(K)$. According to ref. 3, proposition 7, two groups in extensions with the same $\phi(K)$ are isomorphic if and only if it is possible to find representatives of their equivalence classes with factor systems m and $\bar{m}$ respectively, an automorphism ω of K and an element χ of the normalizer $N_{\phi(K)}$ of $\phi(K)$ in $GL(n+1, Z)$ such that

$$\bar{m}(\omega\alpha, \omega\beta) = \chi m(\alpha, \beta) \qquad (\forall \alpha, \beta \in K),$$

$$\chi\phi(\alpha)\, a = \phi(\omega\alpha)\, \chi a \qquad (\forall \alpha \in K, \forall a \in Z^{n+1}).$$

Here we state another equivalent criterion applicable in the case the groups are given by generators and defining relations as in eq. (3.7).

If ω is an automorphism of K choose for $\omega\alpha_j$ a fixed word

$$\omega\alpha_j = w_j(\alpha_1, \ldots, \alpha_\nu) \qquad (j = 1, \ldots, \nu).$$

Define

$$\bar{\kappa} f_i(\bar{g}_1, \ldots, \bar{g}_r) \overset{\text{def}}{=} \phi_i(w_1(\bar{r}(\alpha_1), \ldots, \bar{r}(\alpha_\nu)), \ldots, w_\nu(\bar{r}(\alpha_1), \ldots, \bar{r}(\alpha_\nu)))$$

where

$$\bar{\kappa}\bar{g}_i = \phi_i(\bar{r}(\alpha_1), \ldots, \bar{r}(\alpha_\nu)) \qquad (i = 1, \ldots, r).$$

Furthermore elements $\pi_i(\alpha_j)(i = 1, \ldots, r; j = 1, \ldots, \nu)$ of the integral group ring ZK are defined by

$$\phi_i(\kappa c(\alpha_1) + r(\alpha_1), \ldots, \kappa c(\alpha_\nu) + r(\alpha_\nu)) = \kappa \sum_{j=1}^{\nu} \pi_i(\alpha_j) \cdot c(\alpha_j) +$$
$$+ \phi_i(r(\alpha_1), \ldots, r(\alpha_\nu)) \qquad (i = 1, \ldots, r).$$

Proposition 7:

G^{n+1} and $\bar{G}^{n+1}$ are isomorphic if and only if there are a) an automorphism ω of K, *b*) an element $\chi \in N_{\phi(K)}$ and c) elements $c(\alpha) \in Z^{n+1}$ such that

$$f_i(\bar{g}_1, \ldots, \bar{g}_r) + \sum_{j=1}^{\nu} \pi_i(\omega\alpha_j) \cdot c(\alpha_j) = \chi g_i \qquad (i = 1, \ldots, r). \tag{3.8}$$

Proof:

a) If in eq. (3.5) ψ is an isomorphism, χ is the restriction of ψ to Z^{n+1} and ω is the induced automorphism one has[3]):

$$\phi(\omega\alpha) = \chi\phi(\alpha)\,\chi^{-1} \qquad (\forall \alpha \in K).$$

Therefore $\chi \in N_{\phi(K)}$. Furthermore ψ operating on the relation

$\phi_i(r(\alpha_1), \ldots, r(\alpha_\nu)) = \kappa g_i$ gives

$$\phi_i(\bar{r}(\omega\alpha_1), \ldots, \bar{r}(\omega\alpha_\nu)) + \bar{\kappa} \sum_{j=1}^{\nu} \pi_i(\omega\alpha_j) \cdot c(\alpha_j) = \bar{\kappa}\chi g_i \qquad (i = 1, \ldots, r).$$

Choose for every $\omega\alpha \in K$ a fixed word w $(\alpha_1, \ldots, \alpha_\nu)$. Then

$$\bar{r}(\omega\alpha) = \bar{\kappa}a(\omega\alpha) + w_\alpha(\bar{r}(\alpha_1), \ldots, \bar{r}(\alpha_\nu))$$

for some $a(\omega\alpha) \in Z^{n+1}$. One may choose $a(\omega\alpha) = 0$ for this choice of words, because this only gives an equivalent group. (Of course for another choice of words $w'_\alpha(\alpha_1, \ldots, \alpha_\nu)$ in general $a(\omega\alpha) \neq 0$; it is well known that only in split extensions one can have representatives $r(\alpha)$ in such a way that

$$r(\alpha\beta) = r(\alpha) + r(\beta) \quad \text{for all} \quad \alpha, \beta \in K).$$

Doing this one obtains relation (3.8).

b) On the other hand, if ω and χ are given in such a way that eq. (3.8) is correct, one may define an isomorphism ψ: $G^{n+1} \to \bar{\bar{G}}^{n+1}$, where $\bar{G}^{n+1}$ and $\bar{\bar{G}}^{n+1}$ appear in equivalent extensions, by

$$\psi(\kappa a + r(\alpha)) = \bar{\kappa}\,\chi a + \bar{\kappa}c(\alpha) + \bar{r}(\omega\alpha) \quad (\forall a \in Z^{n+1},\ \forall \alpha \in K,\ c(\alpha) \in Z^{n+1}).$$

Then G^{n+1} and $\bar{G}^{n+1}$ are isomorphic.

So from a set of non-equivalent extensions for a given $\phi(K)$ one can determine the non-isomorphic ones using proposition 7.

4. *Relativistic crystallographic groups*

A *relativistic space-time group* G^{n+1} is a subgroup of the inhomogeneous Lorentz group $JL(n+1)$ which contains a translation subgroup $U^{n+1} \stackrel{\text{def}}{=} G^{n+1} \cap T^{n+1}$ which is free abelian of rank $n+1$ and which over R generates T^{n+1}. This group U^{n+1} is maximal abelian and normal in G^{n+1}. Analogously to the Euclidean case the *relativistic point group* $K \cong G^{n+1}/U^{n+1}$ is a subgroup of $O(n, 1)$ leaving Λ invariant and G^{n+1} appears in an extension

$$0 \to Z^{n+1} \to G^{n+1} \to K \to 1 \qquad (\phi)$$

of a free abelian group Z^{n+1} of rank $n+1$ by the point group K, ϕ being a monomorphism $K \to GL(n+1, Z)$. One of the most important differences from the Euclidean case is the fact that in general K is not finite. In the present paper, however, we are concerned with space-time groups for which the image in $GIL(n+1, R)$ coincides with the image of a Euclidean space group. Therefore the point groups considered here are always finite.

In Minkowskian space timelike, spacelike and isotropic translations occur (having negative, positive and zero norm respectively). For that reason we need a stronger equivalence relation than plain group isomorphism in order to decide if two relativistic space-time groups may be identified or not.

Hence we define: two space-time groups G^{n+1} and $\bar{G}^{n+1}$ are *isomorphic* if there is a group isomorphism $\psi\colon G^{n+1} \to \bar{G}^{n+1}$ such that for the restriction $\chi_0 = \psi | U^{n+1}$ one has

$$\operatorname{sign}\|\chi_0 t\|^2 = \operatorname{sign}\|t\|^2 \qquad (\forall t \in U^{n+1}). \tag{4.1}$$

Here the sign of the norm of an isotropic vector is zero by definition.

Consider a space-time group G^{n+1} (If not stated otherwise in this section we mean by space-time group always a relativistic space-time group), and the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^{n+1} & \xrightarrow{\kappa} & G^{n+1} & \xrightarrow{\sigma} & K & \longrightarrow & 1 \qquad (\phi) \\ & & {\scriptstyle\lambda}\searrow & & \nearrow{\scriptstyle i} & & & & \\ & & & U^{n+1} & & & & & \end{array} \tag{4.2}$$

Here i is the injection of the normal subgroup U^{n+1} into G^{n+1} and λ is an isomorphism which determines a set of generators of U^{n+1}. If $\varepsilon_1 = (1, 0, \ldots, 0)$, $\varepsilon_2 = (0, 1, \ldots, 0)$, $\ldots$, $\varepsilon_{n+1} = (0, \ldots, 0, 1)$, then a basis $B = (a_1, a_2, \ldots, a_{n+1})$ of the lattice Λ is defined by $\lambda\varepsilon_i = a_i$. For the monomorpism κ one has $\kappa = i \circ \lambda$. The reason why we consider i and λ next to κ is the fact that space-time groups are not uniquely determined by their abstract group structure. To the basis B corresponds a metric tensor $g = g(B)$ with elements

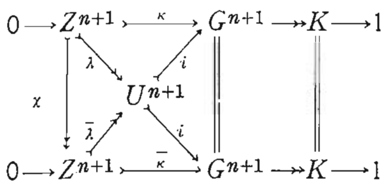
$$g_{ij} = a_i \cdot a_j \qquad (i, j = 1, \ldots, n + 1).$$

The scalar product is given by the indefinite metric of U^{n+1} induced by that of T^{n+1}. A discrete translation group U^{n+1} (or the lattice Λ obtained in Minkowskian space by operating with U^{n+1} on the origin) is defined by its metric tensor up to a homogeneous Lorentz transformation. As K is a group of homogeneous Lorentz transformations leaving invariant the metric tensor one has

$$g(B) = \phi^t(\alpha)\, g(B)\, \phi(\alpha) \qquad (\text{any } \alpha \in K)$$

where $g(B)$ and $\phi(\alpha)$ refer to the same lattice basis B.

Now consider two extensions in which the same space-time group G^{n+1} occurs:



(ϕ)

($\bar{\phi}$)

The elements $\bar{a}_i = \bar{\lambda}\varepsilon_i$ $(i = 1, 2, \ldots, n + 1)$ form another basis $\bar{B}$ of Λ. The monomorphism $\bar{\phi}$: $K \to GL(n + 1, Z)$ is given by:

$$\bar{\phi}(\alpha) = \chi\phi(\alpha)\, \chi^{-1} \qquad (\forall \alpha \in K).$$

The metric tensor corresponding to the basis $\bar{B} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n+1})$ is given by:

$$g(B) = \chi^t g(\bar{B})\, \chi.$$

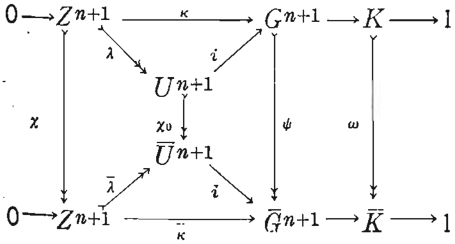
So a space-time group defines a class of pairs $(B, \phi(K))$ with the following equivalence relation: two pairs $(B, \phi(K))$ and $(\bar{B}, \bar{\phi}(K))$ are equivalent if:

$$\bar{\phi}(\alpha) = \chi\phi(\alpha)\, \chi^{-1} \qquad (\forall \alpha \in K),$$

and $\qquad (\chi \in GL(n + 1, Z)) \qquad$ (4.3)

$$g(B) = \chi^t g(\bar{B})\, \chi.$$

Now consider two isomorphic space-time groups G^{n+1} and $\bar{G}^{n+1}$ in the commutative diagram



(ϕ)

(4.4)

($\bar{\phi}$)

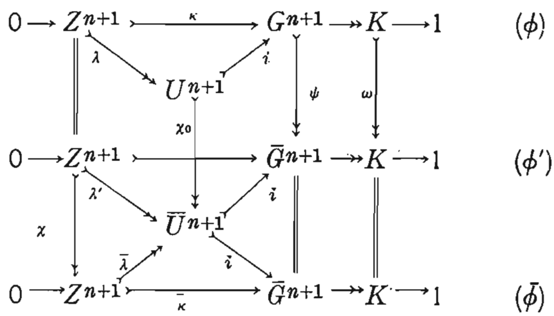
then ψ is an isomorphism such that the restriction χ_0 to U^{n+1} satisfies relation (4.1). i and $\bar{i}$ are the natural injections, so

$$K \cong G^{n+1}/U^{n+1} \cong \bar{G}/\bar{U}^{n+1} \cong \bar{K}.$$

For the arithmetic point groups one has the relation

$$\bar{\phi}(\omega\alpha) = \chi\phi(\alpha)\,\chi^{-1} \qquad (\forall\alpha \in K).$$

Now consider another basis B' of $\bar{\Lambda}$ determined by an isomorphism $\lambda' = \bar{\lambda}\chi$: $Z^{n+1} \to \bar{U}^{n+1}$. Then we may insert a third exact row in (4.4) as follows



(4.5)

The arithmetic point groups are related by

$$\bar{\phi}(\omega\alpha) = \chi\phi'(\omega\alpha)\,\chi^{-1} = \chi\phi(\alpha)\,\chi^{-1}.$$

So $\phi'(\omega\alpha) = \phi(\alpha)$. Hence the isomorphic space-time groups G^{n+1} and $\bar{G}^{n+1}$ may be obtained in (K, ϕ) extensions with the same ϕ.

If the metric tensor for the basis B of Λ is g and that for B' of $\bar{\Lambda}$ is g' (4.1) requires

$$\sum_{i,j=1}^{n+1} p^i p^j g_{ij} \lesseqqgtr 0 \qquad \text{if and only if} \qquad \sum_{i,j=1}^{n+1} p^i p^j g'_{ij} \lesseqqgtr 0 \tag{4.6}$$

$$(\text{any } p^i \in Z).$$

Suppose $g_{kk} \neq 0$ for all $1 \leqslant k \leqslant n+1$. This hypothesis is not restrictive because one can always find a basis without isotropic vectors. Then the images in $\bar{U}^{n+1}$ also have non-zero norm. One may suppose $p^{n+1} \neq 0$. By a change of variables

$$q^i = \frac{p^i}{p^{n+1}} \qquad (i = 1, \ldots, n)$$

eq. (4.6) becomes

$$f(q^1, \ldots, q^n) \stackrel{\text{def}}{=} \sum_{i,j=1}^{n} q^i q^j g_{ij} + \sum_{i=1}^{n} q^i g_{i(n+1)} + g_{(n+1)(n+1)} \gtreqqless 0$$

if and only if:

$$f'(q^1, \ldots, q^n) \stackrel{\text{def}}{=} \sum_{i,j=1}^{n} q^i q^j g'_{ij} + \sum_{i=1}^{n} q^i g'_{i(n+1)} + g'_{(n+1)(n+1)} \gtreqqless 0$$

for all rational q^i. Because the quadratic forms f and f' are zero for the same values of the rational variables they only differ by a real factor k. As the regions of the variables, where the forms are positive, are the same, this k is positive.

So the equivalent space-time groups G^{n+1} and $\bar{G}^{n+1}$ determine together with the isomorphisms λ and $\bar{\lambda}$ two pairs $(B, \phi(K))$ and $(\bar{B}, \bar{\phi}(K))$ respectively such that

$$g(B) = k\chi^t g(\bar{B})\,\chi,$$
$$\bar{\phi}(K) = \chi\phi(K)\,\chi^{-1}, \qquad \chi \in GL(n+1, Z), k > 0. \tag{4.7}$$

Any two pairs which satisfy the relations (4.7) are called equivalent. The equivalence class is called the *relativistic arithmetic crystal class* $\{B, \phi(K)\}$.

From the discussion above and the fact that isomorphisms $\lambda: Z^{n+1} \to U^{n+1}$ and $\bar{\lambda}: Z^{n+1} \to \bar{U}^{n+1}$ determine the same arithmetic crystal class according to the relations (4.3) one has:

Proposition 8:

A space-time group determines a relativistic arithmetic crystal class. Isomorphic space-time groups determine the same relativistic arithmetic crystal class.

To find the arithmetic crystal classes one introduces the concept of geometric crystal class. Two point groups are called relativistic *geometrically equivalent* if they are conjugate subgroups of $O(n, 1)$.

Proposition 9:

If K and $\bar{K}$ are relativistic geometrically equivalent point groups, for each space-time group with point group $\bar{K}$ one has an isomorphic space-time group with point group K.

Proof: Suppose $K = T\bar{K}T^{-1}$ for some $T \in O(n, 1)$. The space-time group $\bar{G}^{n+1}$ occurs in a $(\bar{K}, \phi)$-extension, where $\phi(\bar{K})$ is $\bar{K}$ with respect to a basis $\bar{B}$. Then there is an isomorphic (K, ϕ)-extension for $\phi(K)$ given by K with respect to $B = T\bar{B}$. (Note that $\phi(K) = \phi(\bar{K})$). Because a Lorentz transformation leaves invariant the metric tensor, the translation groups $U^{n+1} \subset G^{n+1}$ and $\bar{U}^{n+1} \subset \bar{G}^{n+1}$ have the same metric tensor with respect to basis B and $\bar{B}$ respectively. Hence G^{n+1} and $\bar{G}^{n+1}$ are relativistic equivalent.

Proposition 10:

It is sufficient to consider the relativistic arithmetic crystal classes for one representative of each relativistic geometric class in order to obtain all relativistic arithmetic crystal classes.

Two lattices Λ and $\bar{\Lambda}$ generated by the translation subgroups $U^{n+1} \subset G^{n+1}$

and $\bar{U}^{n+1} \subset \bar{G}^{n+1}$ of two isomorphic space-time groups have bases with respect to which the metric tensors differ only by a positive real factor.

This means that both lattices have the same holohedry $\phi(H)$ with respect to these bases. Therefore the concept of Bravais class does not play the same important role here as it does in the Euclidean case.

Two lattices Λ and $\bar{\Lambda}$ belong to the same *relativistic Bravais class* if there exists an isomorphism between the generating translation groups U^{n+1} and $\bar{U}^{n+1}$ that maps elements of U^{n+1} on elements of the same kind in $\bar{U}^{n+1}$. (Of course both are isomorphic to Z^{n+1}.) From the foregoing discussion it follows that Λ and $\bar{\Lambda}$ belong to the same relativistic Bravais class if there are bases B of Λ and $\bar{B}$ of $\bar{\Lambda}$ such that for the corresponding metric tensors one has

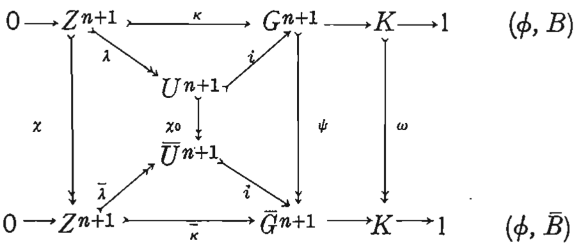
$$g(B) = kg(\bar{B}), \qquad k > 0.$$

Hence a Bravais class may be denoted by a $g(B)$ or by a class $\{B, \phi(H)\}$ because B determines the holohedry $\phi(H)$. Each arithmetic crystal class $\{B, \phi(K)\}$ belongs to a Bravais class $\{B, \phi(H)\}$ and one has $\phi(K) \subseteq \phi(H)$.

In the same way as in the Euclidean case one defines a *relativistic system*. Two lattices belong to the same relativistic system if their geometric holohedries are geometrically equivalent.

Finally one has to find all non-isomorphic space-time groups obtained from a (K, ϕ)-extension and with basis B for one representative of each arithmetic crystal class $\{B, \phi(K)\}$.

Consider two isomorphic space-time groups G^{n+1} and $\bar{G}^{n+1}$. They can always be made to appear in the following commutative diagram



(4.8)

where $kg(\bar{B}) = g(B)$. From $g(B) = k\chi^t g(\bar{B})\, \chi$ one has $\chi \in \phi(H)$. Furthermore because $\phi(\omega\alpha) = \chi\phi(\alpha)\, \chi^{-1}$ one also has $\chi \in N_{\phi(K)}$. Therefore $\chi \in N_{\phi(K)} \cap \phi(H)$.

Proposition 11:

If two space-time groups G^{n+1} and $\bar{G}^{n+1}$ appear in equivalent extensions (4.8) they are isomorphic.

Proof: If the upper and lower rows in (4.8) are equivalent one may choose the isomorphism ψ in such a way that χ is the identity mapping. This means

$$g(B) = kg(\bar{B}).$$

So G^{n+1} and $\bar{G}^{n+1}$ are isomorphic.

As a consequence of propositions 7 and 11 one may formulate the following criterion to distinguish the non-isomorphic space-time groups appearing in diagram (4.8). (We use here the same notation as in section 3.)

Proposition 12:

G^{n+1} and $\bar{G}^{n+1}$ are isomorphic space-time groups if and only if there are automorphisms $\omega \in \operatorname{Aut} K$ and $\chi \in N_{\phi(K)} \cap \phi(H)$ and elements

$$c(\alpha_1), \ldots, c(\alpha_\nu) \in Z^{n+1}$$

such that relation (3.8) is satisfied, *i.e.*:

$$f_i(\bar{g}_1, \ldots, \bar{g}_r) + \sum_{j=1}^{\nu} \pi_i(\omega\alpha_j)\, c(\alpha_j) = \chi g_i \qquad (i = 1, \ldots, r).$$

Proof: As a consequence of proposition 7: if G^{n+1} and $\bar{G}^{n+1}$ have isomorphic abstract structure, one has $\chi \in N_{\phi(K)}$. Besides, as seen above, $\chi \in \phi(H)$.

On the other hand if $\chi \in N_{\phi(K)} \cap \phi(H)$ the groups G^{n+1} and $\bar{G}^{n+1}$ considered in the proof of proposition 7 are not only isomorphic as abstract groups, but even as relativistic space-time groups (Groups obtained from equivalent extensions are isomorphic).

5. *Galilean crystallographic groups*

A Galilean *space-time group* G^{n+1} is a subgroup of $JG(n+1)$ with a translation subgroup $U^{n+1} = G^{n+1} \cap T^{n+1}$ which is free abelian of rank $n+1$ and which over R generates T^{n+1}. A *Galilean point group* K is a subgroup of $G(n+1)$ which leaves an $n+1$-dimensional lattice invariant. As we are only concerned with Galilean space-time groups for which the image in $GIL(n+1, R)$ coincides with the image of a Euclidean space group, these Galilean space-time groups are affine conjugate if and only if they have isomorphic group structure. However, in Galilean space one can distinguish two types of vectors: space-type ($x^{n+1} = 0$) and velocity-type ($x^{n+1} \neq 0$). Therefore we call two Galilean space-time groups isomorphic if there is a group isomorphism between them such that all translation elements of one group are mapped on translation elements of the same type of the other one.

Denoting by Λ the lattice generated by U^{n+1} from a given origin and by $\underline{R}^n$ the hyperplane $x^{n+1} = 0$, we define

$$\lambda \overset{\text{def}}{=} \Lambda \cap \underline{R}^n.$$

Proposition 13:

λ is a l-dimensional lattice for certain $l (0 \leqslant l \leqslant n)$. This is denoted by λ_l.

Proof:

i) λ_l is discrete, Λ being a lattice;

ii) if $r_1, r_2 \in \lambda_l$, then $n_1 r_1 + n_2 r_2 \in \lambda_l$ for any $n_1, n_2 \in Z$; consequently λ_l is a module over Z;

iii) λ_l is generated by at most n basis vectors (which can be chosen as basis vectors of Λ).

It is always possible to choose a basis $a_1, \ldots, a_{n+1}$ of Λ in such a way that $a_1, \ldots, a_l$ are basis vectors of λ_l. This is called a *standard basis* of Λ. If, for one choice of a standard basis, G^{n+1} determines the arithmetic point group $\phi(K)$, then for another choice one has

$$\bar{\phi}(\alpha) = \chi\phi(\alpha)\,\chi^{-1} \qquad (\forall \alpha \in K) \tag{5.1}$$

where

$$\chi = \begin{pmatrix} \chi_1 & \chi_2 \\ 0 & \chi_3 \end{pmatrix} \tag{5.2}$$

and $\chi \in GL(n+1, Z)$, $\chi_1 \in GL(l, Z)$, $\chi_3 \in GL(n+1-l, Z)$. So a Galilean space-time group determines on a standard basis the dimension l of $\underline{R}^n \cap \Lambda$ and a $\phi(K)$ up to conjugation by an element χ (5.2).

Two pairs $[\bar{\phi}(K), \bar{l}]$ and $[\phi(K), l]$ are equivalent if

i) $l = \bar{l}$,

ii) an element $\chi \in GL(n+1, Z)$ of the form (5.2) exists, such that (5.1) is valid.

A *Galilean arithmetic crystal class* is an equivalence class of pairs $[\phi(K), l]$. Then two isomorphic Galilean space-time groups determine the same Galilean arithmetic crystal class. On the other hand, if $[\phi(K), l]$ and $[\bar{\phi}(K), \bar{l}]$ are in the same Galilean arithmetic crystal class, for each Galilean space-time group G^{n+1} in a (K, ϕ)-extension, there is an isomorphic one $\bar{G}^{n+1}$ in a $(K, \bar{\phi})$-extension.

Two point groups are in the same *Galilean geometric crystal class* if they are conjugate subgroups of $G(n+1)$. Again (*cf.* proposition 10) to obtain all Galilean arithmetic crystal classes it is sufficient to consider one representative K of each geometric crystal class. For, suppose $K = T\bar{K}T^{-1}$ for some $T \in G(n+1)$ and let $\bar{K}\bar{\Lambda} = \bar{\Lambda}$, then $KT\bar{\Lambda} = T\bar{\Lambda} \overset{\text{def}}{=} \Lambda$. If $\bar{B}$ is a standard basis of $\bar{\Lambda}$, and $\dim(\underline{R}^n \cap \bar{\Lambda}) = l$, then $B = T\bar{B}$ is a standard basis of Λ and $\dim(\underline{R}^n \cap \Lambda) = l$, because $G(n+1)\cdot\underline{R}^n = \underline{R}^n$.

A lattice Λ determines an *arithmetic holohedry*, *i.e.* the Galilean arithmetic crystal class of the pair $[\phi(H), l]$, when H is the maximal point group leaving Λ invariant. Two lattices belong to the same *Galilean Bravais class* if and only if they determine the same arithmetic holohedry.

Definition: a *split lattice* is an $n+1$-dimensional lattice for which a standard basis may be chosen with $l = n$ and with a_{n+1} along the x^{n+1}-axis.

Proposition 14:

In each Galilean Bravais class for which $l = \dim(\underline{R}^n \cap \Lambda) = n$ there is a split lattice.

Proof: As $l = n$ one has for Λ a standard basis with $a_1, \ldots, a_n \in \underline{R}^n$ and $a_{n+1} = (r_{n+1}, t_{n+1})$. Define

$$\gamma(v) = \begin{pmatrix} \mathbb{1}_n & -v \\ 0\ldots0 & 1 \end{pmatrix}$$

where r_{n+1} is the column vector with components $x^1_{n+1}, \ldots, x^n_{n+1}$ and $v = r_{n+1}/t_{n+1}$. Then $\bar{\Lambda} \overset{\text{def}}{=} \gamma(v)\, \Lambda$ is a split lattice, because $\bar{\Lambda}$ admits the standard basis:

$$\bar{a}_i = a_i \in \underline{R}^n \qquad (i = 1, \ldots, n),$$

$$\bar{a}_{n+1} = [0, t_{n+1}].$$

Λ and $\bar{\Lambda}$ being obtained from each other by a homogeneous Galilean transformation, they belong to the same Galilean Bravais class.

We have restricted ourselves here to space-time groups with a finite point group. However, in the general case one may state the following propositions.

Proposition 15:

The holohedry of a Galilean lattice Λ with $l = n$ contains a free abelian subgroup of rank n.

Proof: Because of proposition 14 one may consider a split lattice in the same Bravais class. Denote the Galilean geometric holohedry by H and suppose

$$\gamma(v) = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} \in H. \tag{5.3}$$

Then in $\underline{R}^n$ one has: $R\lambda_l = \lambda_l$. So, denoting the Euclidean holohedry of λ_l by h, one has $R \in h$. In the hyperplane $x^{n+1} = t_{n+1}$ one has

$$\gamma(v)[a_{n+1} + \lambda_l] = a_{n+1} + \lambda_l,$$

as these are exactly the points in this hyperplane which is left invariant by a Galilean transformation. Hence $vt_{n+1} \in \lambda_l$. So H has the following elements

i) $\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \qquad (\forall R \in h),$

ii) $\begin{pmatrix} \mathbb{1}_n & r_i/t_{n+1} \\ 0 & 1 \end{pmatrix}$ if $a_i = [r_i, t_i]$ is a basis vector $\qquad (i = 1, \ldots, n),$

iii) $\begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -1 \end{pmatrix}.$

The elements ii) are of infinite order and generate a free abelian group of rank n.

Proposition 16:

The holohedry H of a Galilean lattice Λ with $l < n$ has only elements of finite order.

Proof: Let $a_1, \ldots, a_{n+1}$ with $a_i = [r_i, t_i]$ and $r_i \in \underline{R}^n (i = 1, \ldots, n)$ be a standard basis of Λ. Over Z the vectors $a_1, \ldots, a_l$ generate $\lambda_l = \Lambda \cap \underline{R}^n$, over R they generate the vector space R^l. Choose an orthonormal basis such that $e_1, \ldots, e_l$ generate R^l and $e_1, \ldots, e_n$ generate $\underline{R}^n$, whereas e_{n+1} is along the x^{n+1}-axis. With respect to this basis an element of the holohedry H has the form

$$\gamma = \begin{pmatrix} \mathrm{P} & u \\ 0 & \varepsilon \end{pmatrix}$$

where $\mathrm{P} \in O(n)$, such that $\mathrm{P}\lambda_l = \lambda_l$, u is a n-dimensional column vector and $\varepsilon = \pm 1$. Suppose $\varepsilon = +1$. Consider the hyperplane $t = t_i (i = l+1, \ldots, n+1)$. In this hyperplane γ operates as an element (P, ut_i) of $E(n)$ and (P, ut_i) leaves invariant the point sets $r_i + \lambda_l$ and $r_i + R^l$. Either $r_i \in R^l$ or $r_i \notin R^l$.

i) If $r_i \in R^l$ define v_i in the orthoplement $R_\perp^{n-l}$ of R^l in $\underline{R}^n$ in such a way that $v_i + R^l = r_i + R^l$. Then (P, ut_i) leaves invariant the set $r_i + \lambda_l$ in the hyperplane $v_i + R^l$. So $u \in R^l$ and $\mathrm{P}v_i = v_i$.

ii) If $r_i \in R^l$, one has $u \in R^l$ and $v_i = 0$.

So in $R_\perp^{n-l}$ the vectors $v_{l+1}, \ldots, v_{n+1}$ are left invariant by P. These vectors span a d-dimensional space and $d = n - l$, because, if $d < n - l$ the basis vectors $a_1, \ldots, a_{n+1}$ would generate a space of dimension $\leqslant l + d + 1 < n + 1$. Therefore P leaves $R_\perp^{n-l}$ pointwise fixed. So γ has the form

$$\gamma = \begin{pmatrix} \mathrm{P}_l & 0 & u_l \\ 0 & \mathbb{1}_{n-l} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\mathrm{P}_l\lambda_l = \lambda_l$ and u_l an l-dimensional column vector. P_l, being an element of an l-dimensional Euclidean point group, is of finite order m. Hence

$$\gamma^{\mathrm{m}} = \begin{pmatrix} & & w_l \\ & \mathbb{1}_n & 0 \\ 0 & & 1 \end{pmatrix};$$

γ^m leaves invariant $a_{l+1} + \lambda_l$. So $t_{l+1} w_l \in \lambda_l$. In the same way $t_{l+2} w_l \in \lambda_l$. Suppose $w_l \neq 0$. Then $t_{l+1}/t_{l+2} = p/q$ is a rational number $(p, q \in Z)$ and $qt_{l+1} - pt_{l+2} = 0$, *i.e.*:

$$qa_{l+1} - pa_{l+2} \in \underline{R}^n \cap \Lambda = \lambda_l.$$

By hypothesis this not being the case, $w_l = 0$ and therefore γ is of finite order. Because the elements with $\varepsilon = +1$ form a subgroup of index 1 or 2 in the holohedry the proposition has been shown.

To obtain the Galilean arithmetic crystal classes, one takes one representative K of each Galilean geometric class and determines the lattices left invariant by K. One takes one representative $[\phi(H), l]$ of each arithmetic crystal class of the holohedries of these lattices. (We remark that if a lattice occurs with $l = m$, there exist also lattices with $l = m + 1, \ldots, n$). The non-equivalent pairs $[\phi(K), l]$ with $K \subseteq H$ are representatives of the Galilean arithmetic crystal classes.

Now still remains the problem of finding all non-isomorphic Galilean space-time groups for a given $[\phi(K), l]$. Consider the morphism of group extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^{n+1} & \xrightarrow{\kappa} & G^{n+1} & \xrightarrow{\sigma} & K & \longrightarrow & 1 \\ & & \downarrow \chi & & \downarrow \psi & & \downarrow \omega & & \\ 0 & \longrightarrow & Z^{n+1} & \xrightarrow{\bar\kappa} & \bar G^{n+1} & \xrightarrow{\bar\sigma} & K & \longrightarrow & 1 \end{array} \qquad \begin{array}{c} (\phi, l) \\ \\ (\phi, l) \end{array} \tag{5.4}$$

and the commutative diagram

$$\begin{array}{ccccc} Z^{n+1} & \xrightarrow{\lambda} & U^{n+1} & \xrightarrow{i} & G^{n+1} \\ \downarrow \chi & & \downarrow \chi_0 & & \downarrow \psi \\ Z^{n+1} & \xrightarrow{\bar\lambda} & \bar U^{n+1} & \xrightarrow{\bar i} & \bar G^{n+1} \end{array} \qquad \begin{array}{l} \kappa = i \circ \lambda \\ \bar\kappa = \bar i \circ \bar\lambda \end{array} \tag{5.5}$$

If both extensions (5.4) are equivalent, χ can be chosen to be the identity. So G^{n+1} and $\bar G^{n+1}$ are isomorphic as abstract group and the first l basis vectors of Λ corresponding to U^{n+1} are mapped on the first l basis vectors of $\bar\Lambda$ corresponding to $\bar U^{n+1}$. So G^{n+1} and $\bar G^{n+1}$ are isomorphic as Galilean group.

Proposition 17:

The Galilean space-time groups G^{n+1} and $\bar G^{n+1}$ are isomorphic if and only if there are automorphisms $\chi \in N_{\phi(K)}$ and $\omega \in \operatorname{Aut} K$ (5.4) such that χ is of the form (5.2) and relation (3.8) is satisfied:

$$f_i(\bar g_1, \ldots, \bar g_r) + \sum_{j=1}^{\nu} \pi_i(\omega\alpha_j) \cdot c(\alpha_j) = \chi g_i \qquad (i = 1, \ldots, r).$$

Proof: If G^{n+1} and $\bar G^{n+1}$ are isomorphic, there are automorphisms χ and ω, because of proposition 7. Moreover the elements of $\underline{R}^n \cap \Lambda$ are mapped on elements of $\underline{R}^n \cap \bar\Lambda$. This means that χ is of the form (5.2).

On the other hand, if χ and ω exist with the required properties, then G^{n+1} and $\bar G^{n+1}$ are isomorphic as abstract groups and every element of U^{n+1} is mapped by ψ on an element of the same type of $\bar U^{n+1}$ and vice versa.

So both Galilean space-time groups are isomorphic.

6. *Conclusion*

In this part crystallographic groups in Euclidean, Minkowskian, Galilean and so-called product space have been defined and equivalence relations between them stated. In Euclidean space two space groups are isomorphic if they are isomorphic as groups, but because of the existence of various kinds of elements in the other vector spaces, the isomorphism then is more complicated.

In each of the $n + 1$-dimensional spaces mentioned a space-(time) group may be obtained from an extension of a free abelian group of rank $n + 1$ by a point group K with a monomorphism ϕ: $K \to GL(n + 1, Z)$. In Euclidean space every extension of this type with K finite gives rise to a Euclidean space group (ref. 3, p. 557). A comparable proposition is not known for the other spaces, as there infinite point groups may occur. (If K is a finite crystallographic point group, imbedding theorems corresponding to proposition 5 of ref. 3 may be formulated in quite the same way.)

For these reasons only those crystallographic groups are considered here for which the injection in the inhomogeneous linear group coincides with the injection of a generalized magnetic group. For Euclidean space this means that space groups are considered for which the point groups are $n + 1$-reducible over R. Relativistic and Galilean space-time groups are considered only as far as they have finite point groups, *i.e.* in which no Lorentz or Galilean transformations of infinite order occur.

Already for these simpler groups the classification is rather rich. The number of Euclidean and Galilean Bravais classes is finite, but the number of relativistic Bravais classes is infinite with the power of the continuum. In product space the number is enumerably infinite.

The space-time groups, considered here, being isomorphic as groups to a Euclidean space group, the abstract isomorphism classes of these space-time groups may be determined from all non-isomorphic extensions (1.1) where K corresponds (according to fig. 1) to a generalized magnetic point group. The determination of non-isomorphic space groups for given $\phi(K)$ is discussed in ref. 11, the arithmetic crystal classes corresponding to generalized magnetic groups are determined for the case $n + 1 = 4$ in ref. 12.

Acknowledgment. We wish to thank Prof. M. Newman and Prof. J. Mennicke for helpful discussions, and Prof. H. de Vries for critical remarks.

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