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Crystallography in two-dimensional metric spaces*

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Auszug

Die Kristallographie in zweidimensionalen euklidischen, galileischen und minkowskischen Räumen wird definiert und behandelt und zwar derart, daß es nicht nötig ist, zuvor die Art (definit, entartet oder indefinit) des metrischen Tensors, der durch die kristallographischen Transformationen invariant gelassen wird, festzulegen. Es wird gezeigt, daß diese Behandlung zu allgemeinen Gesetzen führt, welche (wenn man sie auf die euklidische Kristallographie beschränkt) ein neues Licht auf die Struktur der gewöhnlichen Kristallographie werfen. Der Begriff des natürlichen Gitters wird eingeführt und erlaubt es, die obenerwähnten Eigenschaften recht einfach zu illustrieren. Auf diese Weise gelangt man zur Einführung der arithmetischen Funktionen $p_k(n)$ und $\Delta p_k(n)$ die für ganze k und n ganzzahlige Werte annehmen. Diese Funktionen gestatten es, alle kristallographischen Transformationen durch eine ganze Zahl n zu parametrisieren; n durchläuft dabei sämtliche ganze Zahlen. Der euklidische Fall liegt bei $|n| < 2$ vor, der galileische bei $|n| = 2$, und bei $|n| > 2$ hat man den minkowskischen (relativistischen) Fall.

Zur Illustration werden die geometrischen Kristallklassen der zweidimensionalen kristallographischen Punktgruppen tabuliert; der Zusammenhang mit den internationalen kristallographischen Bezeichnungen des euklidischen Falles wird angegeben. Die Symmetrien der natürlichen Gitter werden vollständig abgeleitet. Der galileische Fall, obwohl hier eingeschlossen, bedarf noch weiteren Untersuchungen.

Der Zweck dieses Artikels ist es, eine erste synthetische Übersicht über die Prinzipien einer allgemeinen Kristallographie zu geben.

Abstract

Crystallography in two-dimensional Euclidean, Galilean and Minkowskian space is defined and treated by means of a formalism which does not require an a priori specification of the character (definite, singular or indefinite) of the

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metric tensor left invariant by crystallographic transformations. It is shown how the present approach leads to general laws, which (if restricted to the Euclidean case) throw a new light on the structure of usual crystallography. The concept of natural lattice is introduced because it allows a fairly simple illustration of the properties mentioned above. In this way one directly arrives at arithmetic functions $p_k(n)$ and $\Delta p_k(n)$ having integral values for integral k and n. By means of these functions all crystallographic transformations can be expressed and parametrized by the integer n for all $n \in Z$. The Euclidean case is obtained for $|n| < 2$, the Galilean for $|n| = 2$, and the Minkowskian (relativistic) one for $|n| > 2$.

As illustration, the geometric classes of two-dimensional crystallographic point groups are tabulated and the correspondence with the international notation in the Euclidean case is given. Furthermore, the symmetries of natural lattices are briefly discussed and completely set up. Actually the Galilean case ($|n| = 2$), even if included, needs still further investigation. The aim of the paper is to give a first synthetic view of the basic principles underlying crystallography.

1. Introduction

The aim of this paper is to show, from a bird's eye view, how one reaches a better understanding of the usual Euclidean crystallography, by considering it as part of a more general one in which the character of the metric (definite, indefinite or even singular) is not specified a priori, unifying in this way Euclidean, relativistic (or Minkowskian) and Galilean crystallography[1-3].

One of the great advantages of proceeding in this way is that then the Euclidean crystallographic groups appear within the frame of an infinite number of other crystallographic groups, all governed by the same laws, which are expressible by means of arithmetic functions (i.e. functions that take integral values for integer arguments). This basic fact is not so surprising; one knows (considering homogeneous transformations first) that the necessary and sufficient condition for a metric-conserving transformation to be crystallographic is that, with respect to a properly chosen basis, it can be expressed as an n by n matrix with integral entries and determinant ± 1, i.e. as an element of the arithmetic group $GL(n,Z)$ (in the n-dimensional case). *Mutatis mutandis*, the same is true for all the elements of inhomogeneous crystallographic groups (the space groups

[1] T. JANSSEN, A. JANNER and E. ASCHER, Crystallographic groups in space and time. Physica 41 (1969) 541—565; 42 (1969) 41—70 and 71—92.

[2] A. JANNER and E. ASCHER, Bravais classes of two-dimensional relativistic lattices (to appear in Physica).

[3] A. JANNER and E. ASCHER, Relativistic crystallographic point groups (in two dimensions) (to appear in Physica).

for example) after a suitable embedding. This last point, however, needs a more detailed discussion which goes beyond the frame of this paper.

The presence of general laws common to the various metric cases simply reflects the arithmetic character of crystallography. That this is indeed the case is explicitly shown (at least in the two-dimensional case) elsewhere[4]. Here, we restrict ourselves to the remark that there are crystallographic transformations which do not imply any specific character of the metric: consider as very simple examples lattice translations, or total inversion. We call this case the *ametric case*. Other crystallographic transformations (the *metric case*) imply a given type of metric, and this property is attached, so to speak, to particular values of some natural integers, in the same way as such other properties as the order of the symmetry element and its orientation conserving (or not) character.

The relevant crystallographic laws may generally be formulated in a relatively simple way. Their derivation, however, is less simple, at least at the present stage of investigation. Therefore we here restrict our considerations to the two-dimensional case, and some results are quoted without proof. More details can be found in other papers having a more pronounced mathematical character[2,3,5]. Even the two-dimensional case (the simplest non-trivial one) is so rich that a full treatment is not possible within a single paper of reasonable length.

2. Crystallographic restrictions

Consider a two-dimensional vector space V with basis $B = (e_1, e_2)$ and metric tensor $g_{ik} = e_i \cdot e_k$. We use the notation:

$$g(B) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} = (a, b, c), \qquad (2.1)$$

with $a, b, c, \in R$.

Once for all we fix a metric on V as follows:

i) In the Euclidean case [where $\det g(B) > 0$]:

$$g(B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (2.2i)$$

[4] A. Janner and E. Ascher, Crystallographic interpretation of the elements of $GL(2, Z)$ (to be published).

[5] A. Janner and E. Ascher, Symmetry elements of symmorphic space-time groups (in two dimensions) (to be published).

ii) In the Minkowskian or relativistic case [where $\det g(B) < 0$]:

$$g(B) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2 ii}$$

iii) In the Galilean case [where $\det g(B) = 0$]:

$$g(B) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2iii}$$

An orthogonal transformation X of these metric spaces is then defined as a real two-by-two matrix with determinant ± 1, satisfying the relation:

$$X^t g(B) X = g(B). \tag{2.3}$$

In what follows, and until section 5, we restrict our considerations to proper orthogonal transformations (i.e. with $\det X = +1$) and we express these as $X = L$. Improper transformations are taken into account in the four last sections and will be denoted by the letter M (mirrors). A natural parametrization of proper transformations as is well known, is given by:

for rotations: (Euclidean case)
$$L = \pm L(\varphi) = \pm \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

for Lorentz transformations: (Minkowskian case)
$$L = \pm L(\chi) = \pm \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} \tag{2.4}$$

for Galilei transformations:
$$L = \pm L(\psi) = \pm \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}$$

for any φ, χ and $\psi \in R$. In the kinematical interpretation, the two latter transformations relate two inertial frames moving with relative velocity v, and one has:

$$\cosh\chi = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \text{and} \qquad \psi = \frac{v}{v_0} \tag{2.5}$$

where v_0 is some arbitrary (but fixed) velocity and c denotes the velocity of light.

The requirement for a transformation (2.4) to be crystallographic, i.e. to leave a two-dimensional lattice invariant, in general, imposes so-called crystallographic restriction on the transformation L, as well as on the lattice Λ left invariant.

Suppose now that L is a crystallographic transformation; then there is a basis transformation $S: B \rightarrow B'$, i.e. a real non-singular matrix $S \in GL(2, R)$, such that:

$$S^{-1}LS = L' \in SL(2, Z), \tag{2.6}$$

where $SL(2, \mathbf{Z})$ is the subgroup of $GL(2, Z)$ of all 2×2 integral matrices with determinant $+1$. Note that the condition for L' to leave $g(B')$ invariant is equivalent to (2.3):

$$L'^t g(B') L' = g(B') \tag{2.7}$$

where

$$g(B') = S^t g(B) S \tag{2.8}$$

and $B' = BS$; this latter relation means explicity:

$$\left.\begin{aligned} e_1' &= e_1 S_{11} + e_2 S_{21} \\ e_2' &= e_1 S_{12} + e_2 S_{22} \end{aligned}\right\} \quad \text{for} \qquad S = (S_{ik}). \tag{2.9}$$

It follows from (2.6) that a necessary condition for L to be crystallographic is that its trace should be integral:

$$TrL = TrL' = n \in Z. \tag{2.10}$$

In two dimensions, this is actually also a sufficient one. In fact, in the Galilean case, L is crystallographic for every values of ψ. If $\psi \neq 0$ take $S = \begin{pmatrix} 1 & 0 \\ 0 & 1/\psi \end{pmatrix}$.

In the Euclidean and in the relativistic case the trace determines the absolute value of the real parameter φ and χ, respectively, as one has:

$$n = 2\cos\varphi_n, \qquad |n| = 2\cosh\chi_n. \tag{2.11}$$

The values of φ_n are well known, those of χ_n not so interesting. More interesting are the (discrete) values of velocities v_n of the corresponding kinematical interpretation (2.5) given by:

$$v_n = \frac{\sqrt{n^2-4}}{n}\, c \tag{2.12}$$

and indicated for some values of n in Table 1.

Table 1. *Kinematic interpretation of n*

n	3	4	5	6	7	8	9
$\frac{v_n}{c}$	0.745	0.866	0.916	0.942	0.958	0.968	0.975

Table 2 summarizes the general situation.

Table 2. *Two-dimensional proper orthogonal transformations*

$n = Tr\,L$	order	φ, $\|n\| \leqslant 2$	ψ, $\|n\| = 2$	χ, $\|n\| > 2$	type
< -4	∞			$\geqslant 1.56$	Lorentz hyperbolic
-4	∞			1.32	Lorentz hyperbolic
-3	∞			0.96	Lorentz hyperbolic
-2	∞		$\neq 0$		Galilei parabolic
	2	π	0	0	any metric
-1	3	$2\pi/3$			rotation elliptic
0	4	$\pi/2$			rotation elliptic
1	6	$\pi/3$			rotation elliptic
2	1	0	0	0	any metric
	∞		$\neq 0$		Galilei parabolic
3	∞			0.96	Lorentz hyperbolic
4	∞			1.32	Lorentz hyperbolic
> 4	∞			> 1.56	Lorentz hyperbolic

3. Natural lattices

In order to discuss the restrictions imposed on the lattices by crystallographic proper transformations (different from $\pm E$), we first consider a special class of lattices, the so-called natural lattices, which already includes all metric Euclidean lattices but not all relativistic ones: a feature which is also true in the three-dimensional case and seems to be quite generally true. Thus, for the aims of the present paper it is sufficient to consider the class of natural lattices.

Consider a lattice Λ left invariant by an orthogonal transformation L from (2.4). This means that for any lattice vector $x \in \Lambda$, the transformed vector $y = Lx$ also belongs to Λ.

In general, however, x and y are not a basis of Λ.

A lattice is called natural if there is a lattice basis $B' = (e_1', e_2')$ such that:

$$e_1' = Le_1'. \tag{3.1}$$

Referred to the lattice basis B', the transformation L becomes:

$$L' = L_n = \begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix}, \tag{3.2}$$

where $n = TrL \in Z$. The important point is now that (3.2) is true independently of the specific character of the metric left invariant. Note also that L_{-n} is arithmetically equivalent to $-L_n$. In other words, if L_n is e.g. rotation, L_{-n} is a roto-inversion and vice versa.

We recall that two crystallographic transformations are arithmetically equivalent if by a suitable choice of the lattice basis to which they are referred, they are represented by the same integral matrix. So far as lattices are concerned, we may therefore restrict ourselves to natural integers.

The metric tensors left invariant by L_n are given by:

$$g[B'(n)] = \lambda g_n = \lambda(1, n, 1), \qquad \text{with} \qquad \lambda \in R, \lambda \neq 0. \qquad (3.3)$$

One verifies that indeed for $|n| < 2$ the metric is positive definite, for $|n| = 2$ is singular, and for $|n| > 2$ is indefinite. In the Euclidean case ($|n| < 2$), for each value of $|n|$ there is one and only one natural lattice M_n up to a trivial choice of the unit lengths. (Note that $M_{-n} = M_n$). In the relativistic case ($|n| > 3$) for each $|n|$, there are in general two such lattices: a space-like (M_n) if the basis $B'(n)$ consists of space-like vectors and a time-like ($\overline{M}_n$) if $B'(n)$ consists of time-like vectors. Only for $|n| = 3$ is M_3 equivalent with $\overline{M}_3$. Loosely speaking, $\overline{M}_n$ is the image of M_n obtained by interchanging space-like and time-like regions. Of course even for $|n| < 2$, there are lattices of type $\overline{M}_n$ which simply are like the Euclidean ones, but with negative definite metric. The Galilean case is formally obtained from the relativistic one by taking the limit $c \to \infty$; this corresponds to an opening of the light cone, which becomes degenerate and coincides with the space axis. This explains why the natural Galilean lattice can be seen as the limit of a time-like relativistic natural lattice $\overline{M}_2$, whereas in the same limit M_2 degenerates into a one-dimensional lattice. In fact $\overline{M}_2$ has to be seen as a lattice in a (x, v_0t)-plane, not in a (x, ct)-plane. The non-relativistic limiting procedure is somewhat delicate, and we devote therefore the following section to it, showing how L_2 of (3.2) is indeed a Galilean transformation leaving invariant a singular metric $g_2 = (1, 2, 1)$.

The unit cells of the lattices M_n and $\overline{M}_n$ are given in Fig. 1 and 2. One sees that the lattices are rectangular for even n and rhombic for odd n (see Fig. 3). The corresponding bases $\overline{B}(n)$ and metric tensors $g[\overline{B}(n)]$ are:

for even n:

$$\bar{e}_1(n) = e_1$$

$$\bar{e}_2(n) = \frac{\sqrt{|n^2 - 4|}}{2} e_2 \qquad \text{with} \qquad g[\overline{B}(n)] = \begin{pmatrix} 1 & 0 \\ 0 & \frac{4 - n^2}{4} \end{pmatrix}. \qquad (3.4)$$

and for odd n:

$$\bar{e}_1(n) = e_1$$

$$\bar{e}_2(n) = \tfrac{1}{2}\, e_1 + \frac{\sqrt{|n^2 - 4|}}{2}\, e_2 \quad \text{with} \quad g[\bar{B}(n)] = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5 - n^2}{4} \end{pmatrix}. \quad (3.5)$$

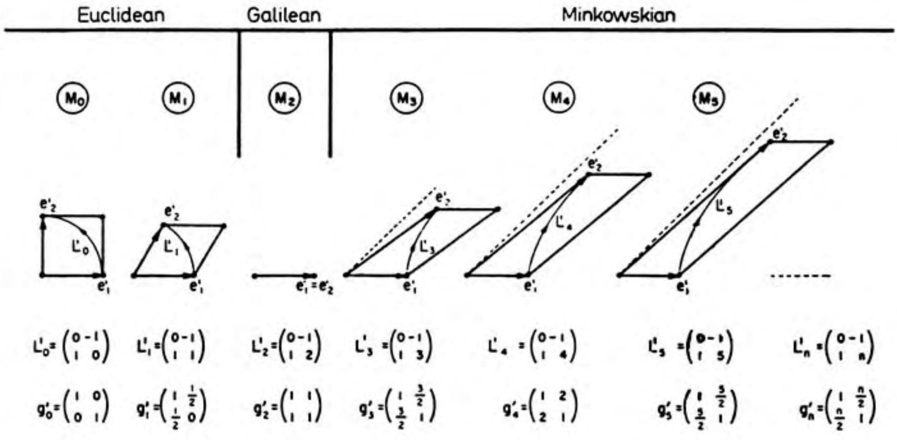


Fig. 1. Space-like natural lattices M_n

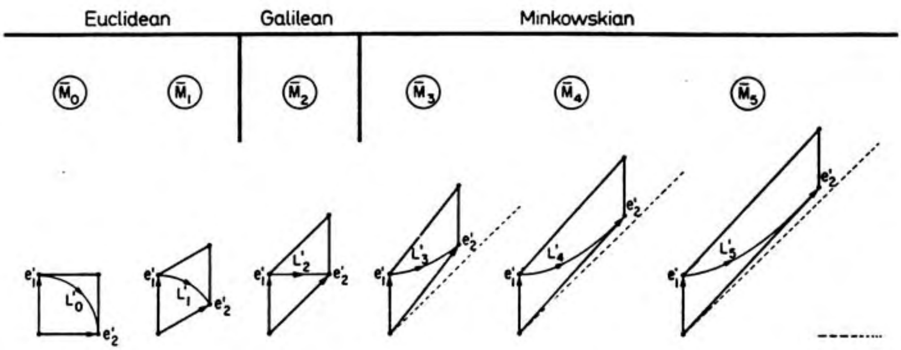


Fig. 2. Time-like natural lattices $\bar{M}_n$

By interchanging e_1 and e_2 one obtains bases for the lattices M_n with metric tensors which are the negative of those indicated in (3.4) and (3.5). For $n = 2$ the lattices become one-dimensional and $\det g[\bar{B}(2)] = 0$. This case needs a more careful treatment.

4. Comments on the Galilean case

We restrict our considerations to Galilean transformations given by:

$$\begin{aligned} x' &= x + vt \\ t' &= t, \end{aligned} \tag{4.1}$$

the other proper transformations being obtained from (4.1) by further applying the total inversion. As is well known, (4.1) is the non-relativistic limit of a Lorentz transformation. Actually one cannot take this limit by simply letting c tend to infinity in (2.5), for one then obtains a trivial transformation: the identity. One has to consider the so-called non-relativistic approximation ($v \ll c$) in which terms like $\frac{v}{c^2}$ and $\frac{v^2}{c^2}$ are neglected. In this approximation $\cosh\chi \cong 1$ and $\sinh\chi \cong \frac{v}{c}$, so that:

$$L(\chi) = L_{\text{gal}} = \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix}, \tag{4.2}$$

which in the (x, ct) plane is indeed the non-relativistic approximation of (4.1).

For a (time-like) natural lattice generated by L_{gal}, we may take the lattice defined by the basis B' given by:

$$\begin{aligned} e_1' &= e_2 \\ e_2' &= L_{\text{gal}}\, e_1' = \frac{v}{c}\, e_1 + e_2, \end{aligned} \tag{4.3}$$

where $B = (e_1, e_2)$ is an orthonormal basis of the (x, ct) plane with the metric tensor as in (2.2*ii*). With respect to B' one has:

$$L'_{\text{gal}} = S^{-1} L_{\text{gal}} S = \begin{pmatrix} 0 & -1 + \frac{v^2}{c^2} \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = L_2 \tag{4.4}$$

where $S = \begin{pmatrix} 0 & \frac{v}{c} \\ 1 & 1 \end{pmatrix}$, so that indeed L_2 can be seen (approximatively) as a crystallographic Galilean transformation in the Minkowskian plane.

We now go over to the (x, v_0t) plane by the basis transformation:

$$E_1 = e_1$$

$$E_2 = \frac{c}{v_0} e_2 \quad \text{for a fixed velocity } v_0 \neq 0 \tag{4.5}$$

and with respect to the basis $B(E_1, E_2), L_{gal}$ becomes:

$$L_{gal} = \begin{pmatrix} 1 & \frac{v}{v_0} \\ 0 & 1 \end{pmatrix} = L(\psi), \qquad \text{for} \qquad \psi = \frac{v}{v_0}. \tag{4.6}$$

The natural lattice $\overline{M}_2$ is then given by:

$$E_1' = E_2$$

$$E_2' = L_{gal} E_1' = \frac{v}{v_0} E_1 + E_2. \tag{4.7}$$

Our expressions do no more depend on c; and now in the Galilean (x, v_0t) plane one has exactly:

$$L'_{gal} = S^{-1} \begin{pmatrix} 1 & \frac{v}{v_0} \\ 0 & 1 \end{pmatrix} S = L_2 \qquad \text{for} \qquad S = \begin{pmatrix} 0 & \frac{v}{v_0} \\ 1 & 1 \end{pmatrix} \tag{4.8}$$

leaving invariant the metric tensor $g_2 = (1, 2, 1) = g[B(E_1', E_2')]$.

It can be seen that one can attach to E_1 and E_2 of (4.7) a metric tensor given by:

$$g[B(E_1, E_2)] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ as in } (2.2\,iii), \tag{4.9}$$

but of course the metric relation given by (4.5) between the Galilean and the Minkowskian planes breaks down in the non-relativistic limit.

Note that the natural lattice M_2 defined as above is rectangular and also obeys the general rule formulated for natural lattices. As discussed elsewhere, Galilean natural lattices are not the only possible lattices in the Galilean plane, but are the only ones invariant with respect to Galilei transformations of infinite order[1].

5. Mirrors

As in (2.4), improper orthogonal transformations leaving (2.2) invariant according to (2.3) can be parametrized by real φ, χ and ψ as follows:

$$\text{in the Euclidean case: } M_\pm = \pm M(\varphi) = \pm \begin{pmatrix} \cos\varphi & -\sin\varphi \\ -\sin\varphi & -\cos\varphi \end{pmatrix}$$

$$\text{in the relativistic case: } M_\pm = \pm M(\chi) = \pm \begin{pmatrix} \cosh\chi & \sinh\chi \\ -\sinh\chi & -\cosh\chi \end{pmatrix} \quad (5.1)$$

$$\text{and in the Galilean case: } M_\pm = \pm M(\psi) = \pm \begin{pmatrix} 1 & \psi \\ 0 & -1 \end{pmatrix}.$$

We call mirrors these transformations M. Their main properties are:

$$\det M = -1, \qquad \operatorname{tr} M = 0 \quad (5.2)$$

and

$$M^2 = E. \quad (5.3)$$

Furthermore, mirrors have eigenvectors μ and ϱ with eigenvalues $+1$ and -1 according to:

$$M_\pm \mu = \pm \mu \quad \text{and} \quad M_\pm \varrho = \mp \varrho. \quad (5.4)$$

In particular for $M(\varphi)$, $M(\chi)$ and $M(\psi)$, these eigenvectors are:

$$\mu(\varphi) = e_1 \cos\frac{\varphi}{2} - e_2 \sin\frac{\varphi}{2}; \qquad \varrho(\varphi) = e_1 \sin\frac{\varphi}{2} + e_2 \cos\frac{\varphi}{2}$$

$$\mu(\chi) = e_1 \cosh\frac{\chi}{2} - e_2 \sinh\frac{\chi}{2}; \qquad \varrho(\chi) = e_1 \sinh\frac{\chi}{2} - e_2 \cosh\frac{\chi}{2} \quad (5.5)$$

$$\mu(\psi) = e_1; \qquad \varrho(\psi) = e_2.$$

Note that in the relativistic case, $\mu(\chi)$ is space like and $\varrho(\chi)$ time like, so that accordingly we may call $M(\chi)$ space like and $-M(\chi)$ time like.

The vectors ϱ and μ are always linearly independent and form a basis which diagonalizes the corresponding mirror, so that for $M_\pm$ as in (5.1) and $B = (\mu, \varrho)$ as in (5.5), one has:

$$M_\pm = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (5.6)$$

showing that any mirror is a crystallographic transformation. Now to turn the other way round (and as discussed in[4]), any element M of $GL(2,Z)$ satisfying (5.2) is of order two and can be interpreted as a mirror leaving lattices invariant in all three different metric spaces. Thus M as well as $\pm E$ is an ametric crystallographic transformation. In fact, any mirror $M \in GL(2,Z)$ belongs either to the rectangular arithmetic class $\{M_R\}$ or to the rhombic one $\{M_D\}$ with representatives:

$$M_R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad M_D = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}. \tag{5.7}$$

These leave invariant the respective metric tensors:

$$G_R = (a,0,c) \quad \text{and} \quad G_D = (a,a,c), \ \forall a, c \in R. \tag{5.8}$$

Choosing various values for the parameters a and c, rectangular or rhombic lattices can be defined in the Euclidean, Galilean and Minkowskian planes, which are left invariant by the given mirror M.

It is worthwhile to remark that a given traceless improper element of $GL(2,Z)$ becomes a specific mirror only if one gives the basis to which it is referred. For example, M_R is a space-like mirror and describes a time reflection if referred to the basis (e_1, e_2) of the Minkowskian plane; it is a time-like mirror and describes a space reflection if the basis is (e_2, e_1), and it is no more a Lorentz transformation if referred to $(e_1 - e_2,\ e_1 + e_2)$.

6. The geometric crystal classes

We recall that a point group K is a group of crystallographic transformations leaving a given metric invariant (2.3). Two point groups K_1 and K_2 belong to the same geometric crystal class if, referred to the basis (2.2), they are conjugate subgroups of:

- the group of orthogonal matrices $O(2)$
- the Lorentz group $O(1,1)$, or
- the Galilei group $G(2)$.

The relativistic geometric point groups (or in other words, the relativistic geometric crystal classes) in two dimensions have been derived in[3].

The Euclidean geometric crystal classes are well known (see e.g.[6] or[7]). The Galilean ones are still under investigation. Here we give only, without proof, a complete list of the geometric crystal

[6] M. J. BUERGER, Elementary crystallography. J. Wiley, New York, 1956.

[7] J. J. BURCKHARDT, Die Bewegungsgruppen der Kristallographie. Birkhäuser, Basel, 1947.

classes all together (Table 3). In Table 4 we indicate the correspondence (in the ametric and Euclidean cases) with the international notation. Recall that the ametric cases are common to the various metrices, whereas in the metric cases the point groups are Euclidean for $|n| < 2$, Galilean for $|n| = 2$, and relativistic for $|n| > 2$.

Table 3. *Geometric crystal classes*

	Abstract point groups	Geometric crystal classes	Generators and relations	Remarks
1. Ametric transformations	C_1	1	E	
	C_2	$\bar{1}$	$-E$ $(-E)^2 = E$	1
		r	M $M^2 = E$	
		m	$-M$ $(-M)^2 = E$	
	D_2	mr	$-E, M$ $(-E)^2 = M^2 = E,$ $(-E)M = M(-E)$	
2. Metric transformations	C_∞ (C_3, C_4, C_6)	$\hat{n}$	L_n $(L_{-1})^3 = E;\ (L_0)^4 = E;\ (L_1)^6 = E$	
	$C_\infty \times C_2$	$\hat{n}\,\bar{1}$	$L_n,\ -E$ $L_n(-E) = (-E)L_n,\ (-E)^2 = E$	2
	D_∞ (D_3, D_4, D_6)	$\hat{n}r$	$L_n,\ M$ $(L_{-1})^3 = E;\ (L_0)^4 = E;\ (L_1)^6 = E$ $M^2 = (ML_n)^2 = E$	3
		$\hat{n}m$	$L_n,\ -M$ $(L_{-1})^3 = E;\ (L_0)^4 = E;\ (L_1)^6 = E$ $(-M)^2 = (-ML_n)^2 = E$	
	$D_\infty \times C_2$	$\hat{n}mr$	$L_n,\ -E,\ M$ $(-E)^2 = M^2 = (ML_n)^2 = E$ $L_n(-E) = (-E)L_n,\ (-E)M = M(-E)$	2

[1] In the Euclidean case $r \sim m$.
[2] $n \geqslant 2$, to avoid duplication.
[3] In the Euclidean case $\hat{n}r \sim \hat{n}m$.
There is a geometric crystal class for every value of n.

Table 4. *Correspondence between our general symbols and the international crystallographic notation*
(In the Euclidean case)

General symbol	1	$\bar{1}$	r m	mr	$-\hat{1}$	$\hat{0}$	$\hat{1}$	$-\hat{1}r$ $-\hat{1}m$	$\hat{0}r$ $\hat{0}m$	$\hat{1}r$ $\hat{1}m$
International notation	1	2	m	$2mm$	3	4	6	$3m$	$4mm$	$6mm$

In reading Table 3, one has to take into account the fact that there are classes which split in the Galilean and relativistic cases and not in the Euclidean one. For example, r and m, from the Euclidean point of view, belong to the single class m. Actually one may consider also e.g. $\hat{0}$ and $\hat{0}\bar{1}$ as degenerating into a single class; in such cases we simply have limited the range of the possible values of n.

A set of representatives for the generators appearing in Table 3 is given in Table 5.

Table 5. *Representatives of the generators appearing in Table 3, and corresponding metric tensors left invariant*

	Ametric case		Metric case	
Generators	E	M	L_n	M
Representatives	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & n\end{pmatrix}$	$\begin{pmatrix}1 & n\\ 0 & -1\end{pmatrix}$
Metric tensors	(a,b,c)	$(a,0,c)$, $a>0$	(a, na, a), $a>0$	

For $a,b,c \in R$ and any $n \in Z$. The choice $a>0$ gives good correspondence with Table 3.

7. Arithmetic functions $p_k(n)$ and $\Delta p_k(n)$

Knowing the symmetries of natural lattices M_n, the determination of the symmetry elements of all symmorphic inhomogeneous groups leaving these lattices invariant is an easy task. The result is a characterization of these symmetries in terms of arithmetic functions $p_k(n)$ and $\Delta p_k(n)$ defined below, valid for all integral values of n, so that, again, the well-known Euclidean symmorphic space groups obtained for $|n| < 2$ appear embedded in a more general frame. The same work can be done also for the non-symmorphic space groups.

In this paper, only the main results are mentioned and generally without proof. A more detailed treatment is given elsewhere[2,5].

The power of L_n [see (3.2)] can be expressed as:

$$L_n^k = \begin{pmatrix} -p_{k-1}(n) & -p_k(n) \\ p_k(n) & p_{k+1}(n) \end{pmatrix}; \quad \forall n, k \in Z \tag{7.1}$$

where $p_k(n)$ is defined by the recurrence relation

$$p_{k+1}(n) = np_k(n) - p_{k-1}(n) \tag{7.2}$$

and the initial values:

$$p_0(n) = 0 \quad \text{and} \quad p_1(n) = 1; \quad \forall n \in Z. \tag{7.3}$$

One finds:

$$p_{-k}(n) = -p_k(n) \tag{7.4}$$

and

$$p_k(-n) = (-1)^{k+1} p_k(n). \tag{7.5}$$

Note that, in agreement with the data of Table 2, $p_k(-1)$, as a function of k ($k \in Z$), is given by a sequence of integers with period three, $p_k(0)$ with period four, and $p_k(1)$ with period six, whereas for $n \geqslant 2$, $p_k(n)$ is a monotone, increasing function of k (see Table 6).

Another important arithmetic function is $\Delta p_k(n)$ defined by:

$$\Delta p_k(n) = Tr L_n^k = p_{k+1}(n) - p_{k-1}(n), \quad \forall n, k \in Z \tag{7.6}$$

which also obeys the recurrence relation:

$$\Delta p_{k+1}(n) = n \Delta p_k(n) - \Delta p_{k-1}(n), \quad \forall n, k \in Z \tag{7.7}$$

and has initial values:

$$\Delta p_0(n) = 2, \quad \Delta p_1(n) = n. \quad \forall n \in Z. \tag{7.8}$$

One finds:

$$\Delta p_{-k}(n) = \Delta p_k(n) \tag{7.9}$$

and

$$\Delta p_k(-n) = (-1)^k \Delta p_k(n). \tag{7.10}$$

Let us also indicate the important relation between $p_k(n)$ and $\Delta p_k(n)$:

$$\Delta p_k^2(n) - 4 = (n^2 - 4)\, p_k^2(n). \tag{7.11}$$

For the proof of (7.11), one uses the fact that $\det L_n^k = 1$, which is equivalent to the relation:

$$p_k^2(n) - p_{k+1}(n)\, p_{k+1}(n) = 1. \tag{7.12}$$

Table 6. *The arithmetic function $p_k(n)$*

k	n								
	-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1	1
2	− 4	− 3	− 2	− 1	0	1	2	3	4
3	15	8	3	0	− 1	0	3	8	15
4	− 56	− 21	− 4	1	0	− 1	4	21	56
5	209	55	5	− 1	1	− 1	5	55	209
6	− 780	− 144	− 6	0	0	0	6	144	780
7	2,911	377	7	1	− 1	1	7	377	2,911
8	− 10,864	− 987	− 8	− 1	0	1	8	987	10,864
9	40,545	2,584	9	0	1	0	9	2,584	40,545
10	− 151,316	− 6,765	− 10	1	0	− 1	10	6,765	151,316
11	564,719	17,711	11	− 1	− 1	− 1	11	17,711	564,719
12	− 1,207,560	−46,368	− 12	0	0	0	12	46,368	1,207,560

Table 7. *The arithmetic function $\Delta p_k(n)$*

k	n								
	-4	-3	-2	-1	0	1	2	3	4
1	− 4	− 3	− 2	− 1	0	1	2	3	4
2	14	7	2	− 1	− 2	− 1	2	7	14
3	− 52	− 18	− 2	2	0	− 2	2	18	52
4	194	47	2	− 1	2	− 1	2	47	194
5	− 724	− 123	− 2	− 1	0	1	2	123	724
6	2,702	322	2	2	− 2	2	2	322	2,702
7	− 10,084	− 843	− 2	− 1	0	1	2	843	10,084
8	37,634	2,207	2	− 1	2	− 1	2	2,207	37,634
9	− 140,452	− 5,778	− 2	2	0	− 2	2	5,778	140,452
10	524,174	15,127	2	− 1	− 2	− 1	2	15,127	524,174
11	− 1,956,244	39,603	− 2	− 1	0	1	2	39,603	1,956,244
12	7,300,802	103,682	2	2	2	2	2	103,682	7,300,802

The functions $p_k(n)$ and $\Delta p_k(n)$ have many other interesting properties, but within the scope of this paper we cannot deal with further details. The values of these two functions are reported for low values of k and n in Tables 6 and 7.

Note in particular that

$$p_k(2) = k, \qquad \Delta p_k(2) = 2, \tag{7.13}$$

which corresponds to the parabolic case. The periods of $p_k(n)$ and $\Delta p_k(n)$ are boxed-in in the two tables.

8. Symmetries of natural lattices

We exclude from our discussion the somewhat pathological case of the Galilean natural lattice ($n = 2$). Actually the formulae that we shall give below for the lattices M_n are still true in the case $n = 2$, but require a careful limiting procedure in order to extract the relevant information from the often singular expressions. This cannot be done in the sketchy way we are forced to adopt here.

As crystallographic coordinate system (see e.g.[6] for this concept), with respect to which the symmetries of natural lattices M_n ($n \neq 2$) are discussed, we adopt the orthogonal axis in the e_1 and e_2 directions, where e_1 and e_2 are given as in (2.1) and (2.2).

We then know from (3.4) and (3.5) that the unit cell of M_n is a rectangular one for even n ($n = 2\nu$) with basis $\bar{B}(n)$ and metric tensor given by:

$$\begin{aligned} \bar{e}_1(2\nu) &= e_1 \\ \bar{e}_2(2\nu) &= \sqrt{|\nu^2 - 1|}\, e_2 \end{aligned} \qquad \text{and} \qquad g[\bar{B}(2\nu)] = \begin{pmatrix} 1 & 0 \\ 0 & 1-\nu^2 \end{pmatrix}, \ \forall \nu \in Z; \quad (8.1)$$

there is only one lattice point per unit cell. For odd n ($n = 2\nu + 1$), however, the unit cell is centered rectangular, thus no more primitive, with basis $\tilde{B}(n)$ and metric tensors:

$$\begin{aligned} \tilde{e}_1(2\nu + 1) &= e_1 \\ \tilde{e}_2(2\nu + 1) &= \sqrt{|4\nu^2 - 4\nu - 3|}\, e_2 \end{aligned}$$

$$\text{and} \qquad g[\tilde{B}(2\nu + 1)] = \begin{pmatrix} 1 & 0 \\ 0 & 3 - 4\nu^2 - 4 \end{pmatrix}, \ \forall \nu \in Z. \quad (8.2)$$

In what follows we systematically use bars for elements referred to primitive rectangular lattices, and tilde signs for elements referred to centered rectangular lattices (see Fig. 3). Thus the proper orthogonal transformations L of (2.4) and (3.2) leaving M_n invariant are given by:

$$\bar{L}_{2\nu} = \begin{pmatrix} \nu & \nu^2 - 1 \\ 1 & \nu \end{pmatrix} \qquad \text{if referred to the basis (8.1)}$$

and (8.3)

$$\tilde{L}_{2\nu+1} = \tfrac{1}{2}\begin{pmatrix} 2\nu + 1 & 4\nu^2 + 2\nu - 3 \\ 1 & 2\nu + 1 \end{pmatrix} \qquad \text{if referred to the basis (8.2).}$$

Note that $\tilde{L}_{2\nu+1}$ is no more a matrix with integral entries because the unit cell adopted is not a primitive one.

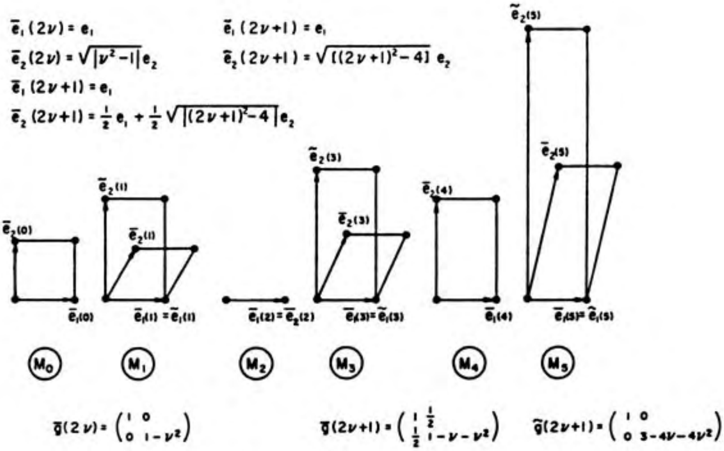


Fig. 3. Natural lattices M_n

We now combine elements of the holohedry H_n of M_n (i.e. the largest point group leaving M_n invariant) with lattice translations (also called primitive translations). Suppose $L \in H_n$; actually L stands for $L(0)$, where 0 is the origin, i.e. the fixpoint of the homogeneous transformation L. This transformation, followed by a lattice translation $t_j \in M_n$, is equivalent with the transformation $L(A_j)$ around a displaced fixpoint A_j (see e.g.[6], page 71 for the Euclidean case). As $L(A_j)$ is the result of the combination of two lattice symmetries, it also leaves the lattice M_n invariant. In other words, A_j is a point with point symmetry L. The set of all A_j having point symmetry

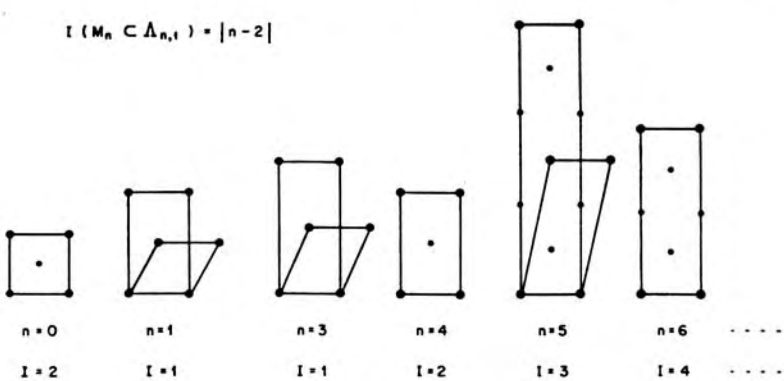


Fig. 4. Lattices $\Lambda_{n,1}$ (point symmetries L_n of M_n)

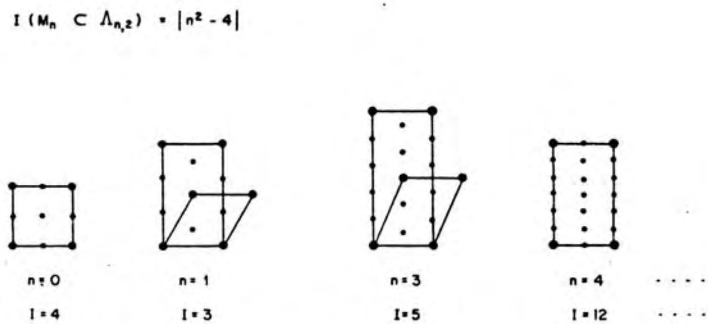


Fig. 5. Lattices $\Lambda_{n,2}$ (point symmetries L_n^2 of M_n)

L_n^k forms a lattice $\Lambda_{n,k}$ that has the lattice M_n as a sublattice. Note that $\Lambda_{n,k}$ is left invariant by L_n^k (see Figs. 4 and 5).

More precisely:

$$L_n^k(A_j)\,\Lambda_{n,k} = \Lambda_{n,k}, \quad \forall\, A_j \in \Lambda_{n,k}. \tag{8.4}$$

The index I of M_n in $\Lambda_{n,k}$ is given by:

$$I(M_n \subset \Lambda_{n,k}) = |\Delta p_k(n) - 2|. \tag{8.5}$$

This means that the primitive cell of M_n contains $|\Delta p_k(n) - 2|$ points A_j having the point symmetry L_n^k.

For *odd* n ($n = 2\nu + 1$), all lattices $\Lambda_{n,k}$ are centered rectangular, with basis vectors given by:

for $k = 2\mu + 1$:

$$\tilde{e}_{1,2\mu+1}(n) = \frac{1}{p_\mu(n) + p_{\mu+1}(n)}\,\tilde{e}_1(n), \qquad \forall\, k = 2\mu + 1,\ \mu \in Z$$

$$\tilde{e}_{2,2\mu+1}(n) = \frac{1}{(n-2)\,[p_\mu(n) + p_{\mu+1}(n)]}\,\tilde{e}_2(n), \quad \forall\, n = 2\nu + 1,\ \nu \in Z \tag{8.6a}$$

and for $k = 2\mu$:

$$\tilde{e}_{1,2\mu}(n) = \frac{1}{p_\mu(n)}\,\tilde{e}_1(n), \qquad \forall\, k = 2\mu,\quad \mu \in Z$$

$$\tilde{e}_{2,2\mu}(n) = \frac{1}{(n^2-4)\,p_\mu(n)}\,\tilde{e}_2(n), \quad \forall\, n = 2\nu + 1,\ \nu \in Z \tag{8.6b}$$

where $\tilde{e}_1(n)$ and $\tilde{e}_2(n)$ are given by (8.2).

Fore *even* n, the lattices $\Lambda_{n,k}$ are (simple) rectangular if k is even, and centered rectangular if k is odd.

The corresponding basis vectors for odd k are:

$$\bar{e}_{1,2\mu+1}(n) = \frac{1}{p_\mu(n) + p_{\mu+1}(n)}\,\bar{e}_1(n), \qquad \forall k = 2\mu+1, \quad \mu \in Z$$

$$\bar{e}_{2,2\mu+1}(n) = \frac{1}{(n-2)\,[p_\mu(n) + p_{\mu+1}(n)]}\,\bar{e}_2(n), \quad \forall n \in 2\nu, \quad \nu \in Z \tag{8.7a}$$

and for even k:

$$\bar{e}_{1,2\mu}(n) = \frac{1}{2\,p_\mu(n)}\,\bar{e}_1(n), \qquad \forall k = 2\mu, \quad \mu \in Z$$

$$\bar{e}_{2,2\mu}(n) = \frac{2}{(n^2-4)\,p_\mu(n)}\,\bar{e}_2(n), \quad \forall n = 2\nu, \quad \nu \in Z. \tag{8.7b}$$

In the same way one finds the set of all points B_j having point-symmetry $(-L_n)^k$. This set also forms a lattice denoted by $\Delta_{n,k}$ left invariant by $(-L_n)^k$:

$$(-L_n)^k(B_j)\Delta_{n,k} = \Delta_{n,k} \qquad \forall B_j \in \Delta_{n,k}. \tag{8.8}$$

Of course, from the definition it follows that

$$\Delta_{n,2\mu} = \Lambda_{n,2\mu} \qquad \forall k = 2\mu, \quad \mu \in Z. \tag{8.9}$$

Actually, as $-L_n$ is arithmetically equivalent with L_{-n} (see section 3), one has more generally:

$$\Delta_{n,k} = \Lambda_{-n,k}, \tag{8.10}$$

but it is still useful to restrict n to natural integers and to distinguish between the two lattices. From (8.6), (8.7) and (8.10), and using (7.10), it follows that the index of M_n in $\Delta_{n,2\mu+1}$ is:

$$I(M_n \subset \Delta_{n,2\mu+1}) = |\Delta\, p_{2\mu+1}(n) + 2|, \quad \forall k = 2\mu+1, \quad \mu \in Z \tag{8.11}$$

and that, for any $n \in Z$, the lattices $\Delta_{n,2\mu+1}$ are centered rectangular with basis given by:

for odd n:

$$\bar{e}_{1,2\mu+1}(\bar{n}) = \frac{1}{p_{\mu+1}(n) - p_\mu(n)}\,\bar{e}_1(n), \qquad \forall k = 2\mu+1, \quad \mu \in Z$$

$$\bar{e}_{2,2\mu+1}(\bar{n}) = \frac{1}{(n+2)\,[p_{\mu+1}(n) - p_\mu(n)]}\,\bar{e}_2(n), \quad \forall n = 2\nu+1, \quad \nu \in Z \tag{8.12}$$

and for even n:

$$\bar{e}_{1,2\mu+1}(\bar{n}) = \frac{1}{p_{\mu+1}(n) - p_{\mu}(n)}\, \bar{e}_1(n), \qquad \forall\, k = 2\mu + 1, \quad \mu \in Z$$

(8.13)

$$\bar{e}_{2,2\mu+1}(\bar{n}) = \frac{2}{(n+2)\,[p_{\mu+1}(n) - p_{\mu}(n)]}\, \bar{e}_2(n), \qquad \forall\, n = 2\nu, \qquad \nu \in Z\,.$$

where $\bar{e}_1(n)$, $\bar{e}_2(n)$ and $\bar{e}_1(n)$, $\bar{e}_2(n)$ are given by (8.2) and (8.1), respectively (see Fig. 6).

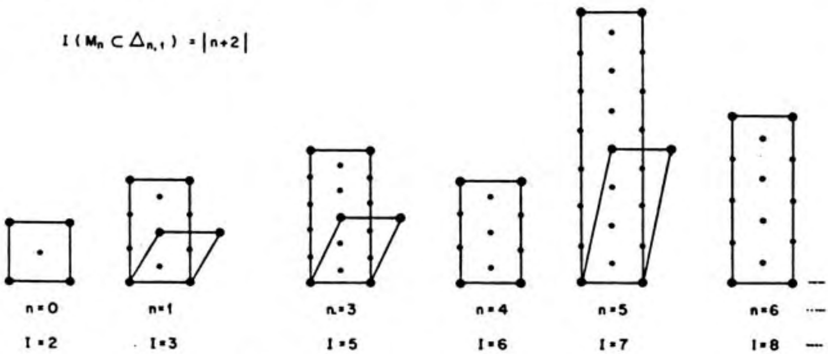


Fig. 6. Lattices $\Delta_{n,1}$ (point symmetries $-L_n$ of M_n)

Combining now improper elements of the holohedries H_n, i.e mirrors through the origin of M_n, with lattice translations, one obtains parallelly displaced mirrors or glides (see e.g.[6] for the Euclidean case). Recall that glides are mirrors with a parallel (non-primitive) component such that two successive glide operations are equivalent to a lattice translation.

As discussed in section 5, mirrors are characterized by the line left invariant: the same is true for glides, because the line (as a whole) is invariant under the translational component of the glide. Thus a mirror is uniquely determined by the slope of its invariant line and a point on it. We also have that, in the relativistic case, one has to distinguish between space-like and time-like mirrors according to whether the invariant line lies outside or inside the light cone. (Mirrors on the light cone are limiting singular cases of the actual mirrors of M_n.) In the Euclidean case, of course, the distinction between the two types of mirror has not the same meaning, but the correspondence is easily done if one remembers that the (x, y) axes correspond to the

(x, ct) ones. For simplicity we therefore use the relativistic terminology in the Euclidean case also.

Any space-like mirror of M_n necessarily has the form

$$r_{n,k} = L_n^k M, \qquad \forall k \in Z, \tag{8.14}$$

where M stands for the time-reflection, which is a symmetry of the lattice M_n. We denote by $\varrho_k(n)$ the slope of the mirror $r_{n,k}$ (i.e. more precisely of its invariant line now identified with the mirror) and by $s_{n,k}$, the glide having the same slope.

In the same way, any time-like mirror of M_n has the form:

$$m_{n,k} = -L_n^k M = -r_{n,k}, \qquad \forall k \in Z, \tag{8.15}$$

where $-M$ is the space reflection, also a symmetry of M_n.

The slope of $m_{n,k}$ is denoted by $\mu_k(n)$ and the corresponding glides by $g_{n,k}$. The holohedry H_n of M_n is $\hat{n}mr$ and is generated by:

$$H_n = \hat{n}mr = \{L_n, M, -E)\}, \quad \forall n \in Z, \tag{8.16}$$

according to Table 3, so that all mirrors $r_{n,k}(O_j)$ and $m_{n,k}(O_j)$ for arbitrary $k \in Z$ occur, with invariant lines through any lattice point $O_j \in M_n$. For given n and k, the set of mirrors $r_{n,k}(O_j)$, called a set of equivalent mirrors, is uniquely determined if one knows, for example, the number of intersections of these mirrors with e_1.

Indeed, in consequence of the (discrete) homogeneity of M_n, these mirrors are all equidistant, and their intersections with the e_1 axis a one-dimensional lattice, which has the lattice generated by e_1 as a sublattice of index $d_k(n)$. One finds:
for even k:

$$d_{2\mu}(n) = p_\mu(n), \qquad \forall k = 2\mu, \quad \mu \in Z \tag{8.17a}$$

and for odd k:

$$d_{2\mu+1}(n) = p(n) + p_{\mu+1}(n), \quad \forall k = 2\mu + 1, \quad \mu \in Z. \tag{8.17b}$$

Similarly we denote by $f_k(n)$ the index of the lattice generated by e_1 in the lattice of intersections of all $m_{n,k}(O_j)$ for given n and k and any $O_j \in M_n$. One has:

for odd k:

$$f_{2\mu+1}(n) = p_{\mu+1}(n) - p_\mu(n), \quad \forall k = 2\mu + 1, \quad \mu \in Z \tag{8.18a}$$

and for even k:

$$f_{2\mu}(2\nu) = \tfrac{1}{2}\,\Delta p_\mu(2\nu) \quad \text{for} \quad n = 2\nu,\ \nu \in Z, \quad \forall k = 2\mu,\ \mu \in Z \quad (8.18b)$$

$$f_{2\mu}(2\nu + 1) = \Delta p_\mu(2\nu + 1) \quad \text{for} \quad n = 2\nu + 1, \quad \mu, \nu \in Z. \quad (8.18c)$$

These two sets of equivalent mirrors $r_{n,k}(O_j)$ and $m_{n,k}(O_j)$ are not the only improper symmetries of the lattice M_n.

Table 8. *Representatives of the equivalence classes of improper symmetries of* M_n

Representatives	k	n	P	Q
$r_{n,k}(O)$, $r_{n,k}(P)$	0 (mod 2)	0 (mod 4)	$\frac{1}{2}\bar{e}_2$	$\frac{1}{2}\bar{e}_2$
$m_{n,k}(O)$, $m_{n,k}(Q)$		2 (mod 4)	$\frac{1}{2}\bar{e}_2$	$\frac{1}{2}\bar{e}_1$
$r_{n,k}(O)$, $s_{n,k}(P)$	1 (mod 2)	0 (mod 2)	$\frac{1}{2}\bar{e}_1$	$\frac{1}{2}\bar{e}_1$
$m_{n,k}(O)$, $g_{n,k}(Q)$	0, 2, 3, 5 (mod 6)	1 (mod 2)	$\frac{1}{4}(\tilde{e}_1 + \tilde{e}_2)$	$\frac{1}{4}(\tilde{e}_1 + \tilde{e}_2)$
	1, 4 (mod 6)	1 (mod 2)	$\frac{1}{2}\tilde{e}_1$	$\frac{1}{2}\tilde{e}_1$

$(\bar{e}_1, \bar{e}_2)$ and $(\tilde{e}_1, \tilde{e}_2)$ are the basis indicated in (8.1) and (8.2) respectively.

For any given k (by fixed n) there are two other sets of equivalent mirrors (or of equivalent glides) parallel and alternating with the sets $[r_{n,k}(O_j)]$ and $[m_{n,k}(O_j)]$, respectively, and at half distance between two equivalent mirrors. Therefore these sets of mirrors (or glides) also pass through points $P_{j,k}$ and $Q_{j,k}$, respectively, which form displaced lattices $M_n(P)$ and $M_n(Q)$. The position of the displaced lattices (i.e. their origins P and Q) and the type of improper transformations (mirrors or glides) depend on the specific values of n and k (see Figs. 7 to 14). In every case, the situation is completely

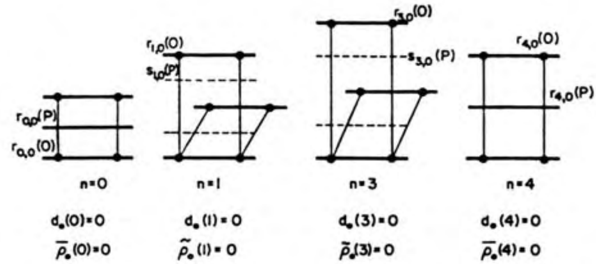
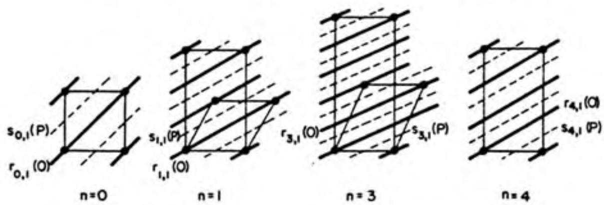


Fig. 7. Space-like mirrors (and glides) of M_n
$k = 0, \quad r_{n,0} = M$

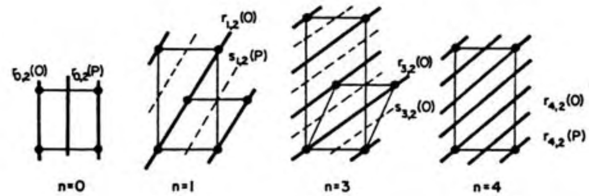
$d_1(n) = 1 \qquad \bar{\rho}_1(2\nu) = \frac{1}{\nu+1} \qquad \tilde{\rho}_1(2\nu+1) = \frac{1}{2\nu+3}$



$d_1(0) = 1$	$d_1(1) = 1$	$d_1(3) = 1$	$d_1(4) = 1$
$\bar{\rho}_1(0) = 1$	$\tilde{\rho}_1(1) = \frac{1}{3}$	$\tilde{\rho}_1(3) = \frac{1}{5}$	$\bar{\rho}_1(4) = \frac{1}{3}$

Fig. 8. Space-like mirrors (and glides) of M_n
$k = 1, \quad r_{n,1} = L_n M$

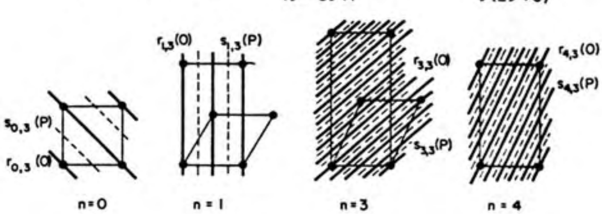
$d_2(n) = 1 \qquad \bar{\rho}_2(2\nu) = \frac{1}{\nu} \qquad \tilde{\rho}_2(2\nu+1) = \frac{1}{2\nu+1}$



$d_2(0) = 1$	$d_2(1) = 1$	$d_2(3) = 1$	$d_2(4) = 1$
$\bar{\rho}_2(0) = \infty$	$\tilde{\rho}_2(1) = 1$	$\tilde{\rho}_2(3) = \frac{1}{3}$	$\bar{\rho}_2(4) = \frac{1}{2}$

Fig. 9. Space-like mirrors (and glides) of M_n
$k = 2, \quad r_{n,2} = L_n^2 M$

$d_3(n) = n+1 \qquad \bar{\rho}_3(2\nu) = \frac{4\nu^2-1}{4\nu^2-3\nu+1} \qquad \tilde{\rho}_3(2\nu+) = \frac{\nu+1}{\nu(2\nu+3)}$



$d_3(0) = 1$	$d_3(1) = 2$	$d_3(3) = 4$	$d_3(4) = 5$
$\bar{\rho}_3(0) = -1$	$\tilde{\rho}_3(1) = \infty$	$\tilde{\rho}_3(3) = \frac{2}{5}$	$\bar{\rho}_3(4) = \frac{5}{9}$

Fig. 10. Space-like mirrors (and glides) of M_n
$k = 3, \quad r_{n,3} = L_n^3 M$

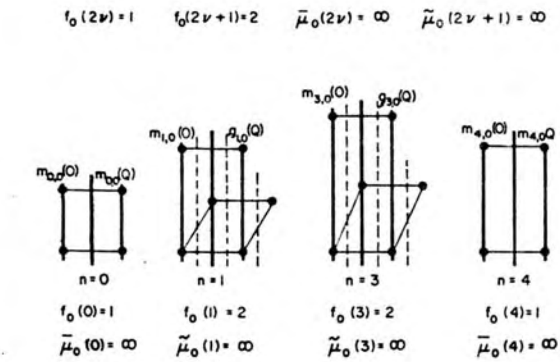


Fig. 11. Time-like mirrors (and glides) of M_n
$k = 0, \quad m_{n,0} = -M$

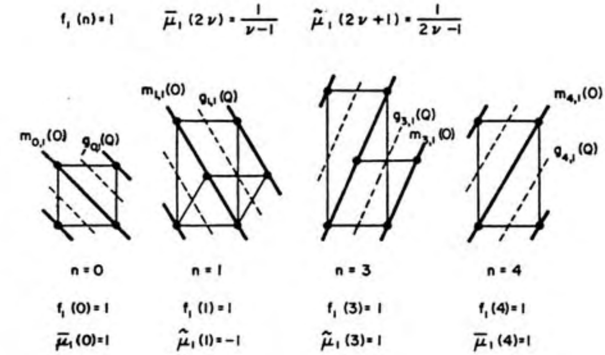


Fig. 12. Time-like mirrors (and glides) of M_n
$k = 1, \quad m_{n,1} = -L_n M$

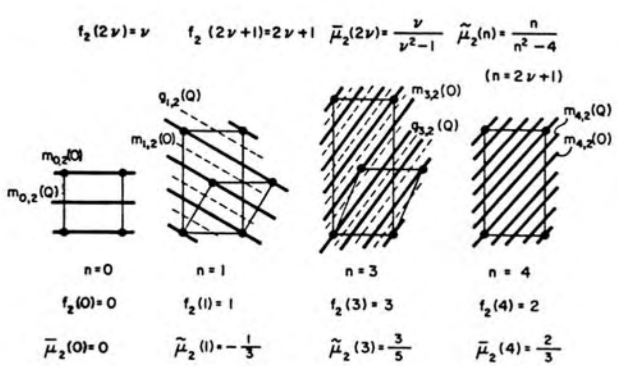


Fig. 13. Time-like mirrors (and glides) of M_n
$k = 2, \quad m_{n,2} = -L_n^2 M$

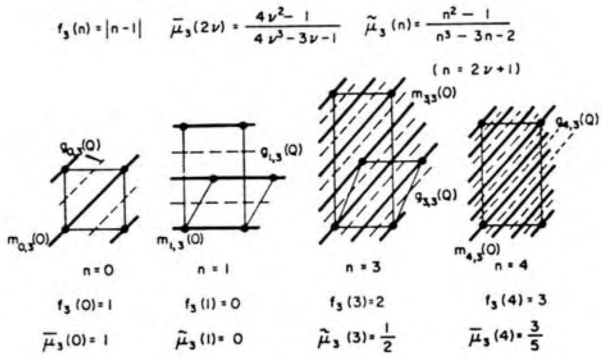


Fig. 14. Time-like mirrors (and glides) of M_n
$k = 3, \quad m_{n,3} = -L_n^3 M$

determined by a set of four inequivalent mirrors (or glides) that are the representatives of the corresponding equivalence classes. The general result is indicated in Table 8.

Finally we also indicate explicitly the symmetries occurring and the slope of the various mirrors and (or) glides.

Suppose first $n = 2\nu$, i.e. even, and a basis of $M_{2\nu}$ given by (8.1). Then with respect to this basis one has:

$$\overline{L}_{2\nu}^{k} = \begin{pmatrix} \nu p_k(2\nu) - p_{k-1}(2\nu) & (\nu^2 - 1) p_k(2\nu) \\ p_k(2\nu) & p_k(2\nu) - p_{k-1}(2\nu) \end{pmatrix} \tag{8.19}$$

and $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore for space-like mirrors:

$$\bar{r}_{2\nu,k} = \begin{pmatrix} \nu p_k(2\nu) - p_{k-1}(2\nu) & (1 - \nu^2) p_k(2\nu) \\ p_k(2\nu) & -\nu p_k(2\nu) + p_{k-1}(2\nu) \end{pmatrix} \tag{8.20}$$

with slope $\varrho_k(n)$ determined from the relation:

$$r_{n,k} \begin{pmatrix} 1 \\ \varrho_k(n) \end{pmatrix} = \begin{pmatrix} 1 \\ \varrho_k(n) \end{pmatrix} \tag{8.21}$$

giving in our case:

$$\bar{\varrho}_k(2\nu) = \frac{p_k(2\nu)}{\nu p_k(2\nu) - p_{k-1}(2\nu) + 1}\,. \tag{8.22}$$

The corresponding formulae for time-like mirrors are

$$\overline{m}_{2\nu,k} = -\bar{r}_{2\nu,k} \tag{8.23}$$

and

$$\overline{\mu}_k(2\nu) = \frac{p_k(2\nu)}{\nu p_k(2\nu) - p_{k-1}(2\nu) - 1} \tag{8.24}$$

for the slope. For $n = 2\nu + 1$, i.e. odd, and a basis of $M_{2\nu+1}$ given by (8.2), one obtains:

$$\tilde{L}_n^k = \tilde{L}_{2\nu+1}^k = \tfrac{1}{2}\begin{pmatrix} \Delta p_k(n) & n\Delta p_k(n) - 4\,p_k(n) \\ p_k(n) & \Delta p_k(n) \end{pmatrix}, \tag{8.25}$$

giving for the space-like mirrors:

$$\tilde{r}_{n,k} = \tilde{r}_{2\nu+1,k} = \tfrac{1}{2}\begin{pmatrix} \Delta p_k(n) & 4\,p_k(n) - n\Delta p_k(n) \\ p_k(n) & -\Delta p_k(n) \end{pmatrix} \tag{8.26}$$

with slope:

$$\tilde{\varrho}_k(2\nu + 1) = \frac{p_k(2\nu + 1)}{\Delta p_k(2\nu + 1) + 2}. \tag{8.27}$$

For time-like mirrors

$$\tilde{m}_{n,k} = -\tilde{r}_{n,k}$$

and the slope is:

$$\tilde{\mu}_k(2\nu + 1) = \frac{p_k(2\nu + 1)}{\Delta p_k(2\nu + 1) - 2}. \tag{8.28}$$

Note that the replacement $r_{n,k} \to m_{n,k}$ entails the replacements $p_k(n) \to -p_k(n)$ and $\Delta p_k(n) \to -\Delta p_k(n)$.

Acknowledgements

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