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RELATIVISTIC CRYSTALLOGRAPHIC POINT GROUPS IN TWO DIMENSIONS

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Synopsis

Two-dimensional relativistic point groups are investigated and various equivalence classes of them are explicitly given. It is shown that every two-dimensional relativistic point group can always be generated by at most three elements (the total inversion, a proper Lorentz transformation of infinite order, and a mirror, *i.e.* an improper Lorentz transformation). There are 7 abstract point groups (C_1, C_2, C_∞, D_2, D_∞, $C_\infty \times C_2$ and $D_\infty \times C_2$) divided into an infinite number of geometric (or R-equivalent) and arithmetic (or Z-equivalent) crystal classes, which are indicated. The relativistic geometric crystal classes (conjugated crystallographic subgroups of the Lorentz group) are also given. All elements of $GL(2, Z)$ that can be interpreted as crystallographic transformations of the Minkowskian metric space are discussed. The rôle of point group symmetry in physical systems having a crystallographic structure in space and time is considered and the importance of point groups for recognizing such systems in nature is underlined.

1. *Introduction.* The present, second step towards a formulation of relativistic crystallography is the natural continuation of the paper on Bravais classes of two-dimensional relativistic lattices[1]).

The physical significance of this research has already been sketched in that paper and, moreover, is still a matter for investigation. Here we should like to mention only one point of view, which may be of importance in the search for physical systems having a symmetry described by space-time groups.

In nature these systems need not be infinite in order to be conveniently described in terms of space-time groups. As in the case of crystals, it is sufficient for such a description to be valid that one such system has a linear dimension L of the order of $10^8\, l$, where l is the length of a typical generator of the lattice translation (time being of course measured as a length by putting $c = 1$). On the macroscopic scale we may characterize the situation by saying that the system is large with respect to the unit cell of the lattice.

Macroscopically, however, physical properties can be considered as invariant with respect to lattice translations. This means that, on the L-scale what we observe are properties of the point group that is isomorphic to the space-time group (or the space group) modulo lattice translations. This situation is well known in crystallography; the theory of crystal forms is based thereon [2]).

In the relativistic case, and if l is submicroscopic, then L may very well be microscopic and on this scale, relativistic point group symmetry could represent the first indication of a finer underlying symmetry pattern. At the other extreme, if l is very large, then L would be of cosmological dimension: actually one of the first applications of relativistic crystallography has been precisely to cosmological models [3,4]).

Of course in nature one has to look primarily for four-dimensional point groups. Those of finite order have already been studied and tabulated [5,6]). The physically most striking case, however, is that where a relativistic point group of infinite order appears as a symmetry group.

A full characterization of these groups appears to be a tremendous task. Yet, on the basis of the experience gathered in the two-dimensional case it is possible to derive a number of families containing an infinite number of four-dimensional point groups, and this knowledge may be sufficient for pursuing the physical investigation.

2. *Crystallographic interpretation of some elements of* $GL(2, Z)$. The interpretation of every element of $GL(2, Z)$ as a crystallographic transformation corresponds to a theory of two-dimensional crystallography formulated independently of the character of the metric of the underlying vector space. Crystallographic transformation has here been considered in the sense of a g-automorphism of the metric space leaving a lattice invariant. We recall that g-automorphisms are either metric conserving linear transformations (automorphisms) or linear transformations reversing the sign of the metric tensor (negautomorphisms). For more details, we refer to ref. 1. In that paper we pointed out that such a general crystallographic programme can indeed be carried out. Here, however, we consider only the indefinite case.

The identity

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the total inversion $-E$ are crystallographic transformations in a trivial way. We have already seen (corollary 1.1. of ref. 1) that every element L of $SL(2, Z)$ with $|\mathrm{tr}\, L| > 2$ can be regarded as a crystallographic Lorentz transformation. Those are the only elements of $SL(2, Z)$ which leave the metric of a relativistic lattice invariant.

Improper elements of $GL(2, Z)$ can be interpreted according to the following two propositions.

Proposition 1. Any element $A \in GL(2, Z)$ with $\det A = -1$ and $\operatorname{tr} A \neq 0$ is an improper negautomorph of a primitive indefinite quadratic form, and therefore leaves a corresponding lattice invariant.

Proof. Let

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and put $|v| = gcd\ (\gamma, \delta - \alpha, -\beta)$. We may suppose $v \neq 0$, because $\gamma = \delta - \alpha = \beta = 0$ and $\alpha\delta - \beta\gamma = -1$ are incompatible. Take the quadratic form

$$\left(\frac{\gamma}{v}, \frac{\delta - \alpha}{v}, \frac{-\beta}{v}\right) \stackrel{\text{Def}}{=} (a, b, c)$$

and choose sign v = sign γ. This quadratic form is clearly primitive and indefinite. The indefiniteness is due to the fact that the discriminant of the quadratic form is positive:

$$d = b^2 - 4ac = \frac{m^2 + 4}{v^2} > 0, \qquad m = \operatorname{tr} A \neq 0. \tag{2.1}$$

Consider the basis $B' = (e'_1, e'_2)$ defined by:

$$\begin{aligned} e'_1 &= \sqrt{a}\, e_1, \\ e'_2 &= \frac{b}{2\sqrt{a}} e_1 + \frac{\sqrt{d}}{2\sqrt{a}} e_2, \end{aligned} \tag{2.2}$$

where e_1 and e_2 are two orthonormal vectors in the Minkowskian space with metric tensor $g(B) = (1, 0, -1)$. Note that in (2.2) a is positive. This means that $\gamma \neq 0$, a consequence of our hypothesis, because for $\gamma = 0$, $\alpha\delta = -1$ and necessarily, $\alpha + \delta = 0$, in contradiction with $\operatorname{tr} A \neq 0$. Now (2.2) defines the metric tensor $g(B')$ of a relativistic primitive lattice Λ. One verifies that

$$A^t g(B')\, A = -g(B'), \tag{2.3}$$

which shows that A is an improper negautomorph of Λ (cf. (3.26) in ref. 1).

Proposition 2. Any element $M \in GL(2, Z)$ with $\det M = -1$ and $\operatorname{tr} M = 0$ is of order two and defines a crystallographic mirror (improper automorph) of an ambiguous lattice.

Proof. Let

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

with $\alpha^2 + \beta\gamma = 1$. It follows that $M^2 = E$. There are two cases:

(i) Suppose $\alpha^2 = 1$; then β or γ, or both, is zero. In proposition 14 of ref. 1 we already showed that matices of the form

$$M_1 = \pm\begin{pmatrix}1 & n\\ 0 & -1\end{pmatrix} \quad \text{or} \quad M_2 = \pm\begin{pmatrix}1 & 0\\ n & -1\end{pmatrix}, \qquad \forall n \in Z \tag{2.4}$$

are arithmetically equivalent to the rectangular mirror

$$M_R = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

if $n \equiv 0 \pmod 2$ and to the rhombic mirror

$$M_D = \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$$

otherwise.

(ii) Suppose now $\alpha^2 \neq 1$. M leaves invariant the lattice vector

$$P = \begin{pmatrix}p_1\\ p_2\end{pmatrix}$$

where $p_1, p_2 \in Z$ are given by the irreducible part of the fraction $-\beta/(\alpha - 1)$:

$$\frac{p_1}{p_2} = \frac{-\beta}{\alpha - 1}, \qquad gcd(p_1, p_2) = 1. \tag{2.5}$$

One knows there are integers q_1, q_2 such that

$$p_1 q_2 - p_2 q_1 = 1. \tag{2.6}$$

Taking

$$S = \begin{pmatrix}p_1 & q_1\\ p_2 & q_2\end{pmatrix} \in SL(2, Z)$$

one finds:

$$S^{-1}MS = \begin{pmatrix}1 & n\\ 0 & -1\end{pmatrix}, \qquad n = 2q_1q_2\alpha + q_2^2\beta - q_1^2\gamma$$

and we are back to the first case.

Now M_R leaves invariant a rectangular lattice $(a, 0, c)$; M_D leaves invariant a rhombic lattice (a, a, c). Clearly this property is independent of the character of the metric expressed by the sign of the discriminant d.

Corollary 2.1. The elements of $GL(2, Z)$ with determinant -1 and vanishing trace belong to two arithmetic classes: the rectangular $\{M_R\}$ and the rhombic class $\{M_D\}$.

3. *Properties of (non-isotropic) primitive lattices.* Primitive lattices have metric tensors $g(B') = (a, b, c)$ with $a, b, c \in Z$, $a > 0$ and $gcd(a, b, c) = 1$; the relation between metric tensor and basis B' generating the lattice is given by (2.2). If no lattice points lie on the light cone, the lattice is non-isotropic and the discriminant $d = b^2 - 4ac$ not a squared integer.

Ambiguous primitive lattices are of type I if left invariant by negautomorphs, otherwise they are of type II (see ref. 1, section 4). Furthermore, the improper automorphs may belong to a single arithmetic class – the rectangular of the rhombic one – and the lattices are then called rectangular (R) or rhombic (D). If the mirrors belong to two different arithmetic classes, the lattice is mixed (RD). We show that this is never the case for a type I lattice.

Proposition 3. All the improper automorphs of a type I lattice Λ belong to a single arithmetic class.

Proof. Any improper automorph of Λ can be written as

$$M = \pm M_0 L_0^k \tag{3.1}$$

with M_0 an arbitrary mirror leaving Λ invariant and L_0 the fundamental Lorentz transformation of primitive $g(B) = (a, b, c)$ (cf. (3.14) in ref. 1). We recall that

$$L_0 = \begin{pmatrix} \frac{1}{2}(n_1 - bu_1) & -cu_1 \\ au_1 & \frac{1}{2}(n_1 + bu_1) \end{pmatrix} \tag{3.2}$$

with (n_1, u_1) the least positive solution of Pell (plus) equation

$$n^2 - du^2 = 4. \tag{3.3}$$

By hypothesis, the corresponding Pell (minus) equation

$$m^2 - dv^2 = -4 \tag{3.4}$$

also has a least positive solution (m_1, v_1) so that

$$L_0 = A_0^2 \qquad \text{with} \qquad A_0 = \begin{pmatrix} \frac{1}{2}(m_1 - bv_1) & -cv_1 \\ av_1 & \frac{1}{2}(m_1 + bv_1) \end{pmatrix} \tag{3.5}$$

is an improper negautomorph. According to the conjugation table (3.5) in ref. 1 we have:

$$M_0 L_0^k = -A_0^{-1} M_0 A_0^k; \tag{3.6}$$

with $M \overset{a}{\sim} -M$, it follows that $M \overset{a}{\sim} M_0$ (where "$\overset{a}{\sim}$" means arithmetic equivalency, cf. (2.9) in ref. 1).

Proposition 4. The product of two improper automorphs M_1 and M_2 of a primitive lattice Λ, not belonging to one and the same arithmetic class, is expressible as:

$$M_1M_2 = \pm L_0^k \quad \text{with} \quad k \equiv 1 \pmod 2, \tag{3.7}$$

where L_0 is given by (3.2) with

$$n_1 \equiv 0 \pmod 2 \qquad \text{and} \qquad u_1 \equiv 1 \pmod 2.$$

We first prove the following lemma:

Lemma 4. If a primitive (A, A, C) is arithmetically equivalent to $(a, 0, c)$, then $A \equiv 0 \pmod 2$ and $C \equiv 1 \pmod 2$.

Proof of the lemma. Consider S such that

$$S^t \begin{pmatrix} A & \frac{A}{2} \\ \frac{A}{2} & C \end{pmatrix} S = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \qquad S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, Z). \tag{3.8}$$

One finds:

$$\begin{aligned} a &= A(\alpha^2 + \alpha\gamma) + C\gamma^2, \\ c &= A(\beta^2 + \beta\delta) + C\delta^2; \\ A[\pm 1 &- 2\alpha(\delta + \beta)] = 2\delta\gamma C. \end{aligned} \tag{3.9}$$

Thus $A \equiv 0 \pmod 2$, and as C is relatively prime to A, $C \equiv 1 \pmod 2$.

Proof of the proposition. Suppose

$$M_1 \overset{a}{\sim} M_R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_2 \overset{a}{\sim} M_D = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

By appropriate choices, $\bar{B}$ or B', of the basis, the metric tensor of Λ takes the rectangular form $g(\bar{B}) = (a, 0, c)$ or the rhombic one $g(B') = (A, A, C)$. Consider a transformation S as above. Then expressing (3.7) with respect to the basis $\bar{B}$, one has:

$$\bar{M}_1\bar{M}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S^{-1} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} S = \begin{pmatrix} 1 + 2\beta\gamma + \gamma\delta & 2\beta\delta + \delta^2 \\ 2\alpha\gamma + \gamma^2 & 1 + 2\beta\gamma + \gamma\delta \end{pmatrix} =$$

$$= \pm \begin{pmatrix} \frac{\Delta p_k(n_1)}{2} & -cu_1 p_k(n_1) \\ au_1 p_k(n_1) & \frac{\Delta p_k(n_1)}{2} \end{pmatrix} = \pm L_0^k,$$

where $p_k(n)$ is defined by

$$p_{k+1}(n) = np_k(n) - p_{k-1}(n), \qquad p_0(n) = 0, \quad p_1(n) = 1, \\ \forall k, n \in Z \tag{3.10}$$

and $\Delta p_k(n) = p_{k+1}(n) - p_{k-1}(n)$. (See (3.14) and (3.15) in ref. 1.) From the lemma and relations (3.9):

$$a \equiv \gamma (\text{mod } 2) \qquad \text{and} \qquad c \equiv \delta \, (\text{mod } 2). \tag{3.11}$$

But we also have:

$$\pm \, a u_1 p_k(n_1) = 2\alpha\gamma + \gamma^2 \qquad \text{and} \qquad \mp c u_1 p_k(n_1) = 2\beta\delta + \delta^2.$$

As a and c are relatively prime, one of them at least is odd so that, together with (3.11), one finds $u_1 \equiv 1$ (mod 2) and $p_k(n_1) \equiv 1$ (mod 2). Furthermore, L_0 being an automorph of a rectangular lattice, $\Delta p_1(n_1) \equiv n_1$ is necessarily even. Using (3.10) for even n_1, it follows from $p_k(n_1)$ being odd that k is also odd. This ends the proof.

Proposition 5. Suppose that the least positive solution (n_1, u_1) of the Pell (plus) equation, $n^2 - du^2 = 4$, is such that

$$n_1 \equiv 0 \, (\text{mod } 2) \qquad \text{and} \qquad u_1 \equiv 1 \, (\text{mod } 2). \tag{3.12}$$

Then for this discriminant d there are no lattices of type I, and any ambiguous lattice is of type II RD.

Proof. It is sufficient to prove that

$$M_R \overset{\mathrm{a}}{\nsim} M_R L_0, \qquad M_R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then necessarily also $M_D \overset{\mathrm{a}}{\nsim} M_D L_0$, and the lattice is of type II RD (according to proposition 3). Consider $g(B) = (a, 0, c)$. Then

$$L_0 = \begin{pmatrix} \nu_1 & -cu_1 \\ au_1 & \nu_1 \end{pmatrix} \qquad \text{with} \qquad 2\nu_1 = n_1 \qquad \text{and} \qquad \nu_1^2 + ac\, u_1^2 = 1.$$

The condition of arithmetic equivalency between M_R and $M_R L_0$ by

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, Z)$$

implies

$$2\beta\gamma = \pm(\nu_1 - 1); \qquad 2\alpha\delta = \pm(\nu_1 + 1) \tag{3.13}$$

and

$$2\beta\alpha = \pm c u_1; \qquad 2\delta\gamma = \pm a u_1. \tag{3.14}$$

If, now, u_1 is odd, then as a and c are relatively prime, condition (3.14) can never be fulfilled.

Corollary 5.1. Condition (3.12) for the least positive solution of the Pell (plus) equation is a necessary and sufficient condition for an ambiguous lattice with this discriminant to be of the sort RD. The lattice is then of type

II. (Note that this is not necessarily true if one considers an arbitrary solution (n, u) instead of the fundamental one.)

Corollary 5.2. If n_1 of proposition 5 is

$$n_1 \equiv 0 \pmod 4, \tag{3.15}$$

then the ambiguous lattices of discriminant d are of type II RD.

Proof. Suppose that the lattice is rectangular. Then with (3.15), relation (3.13) cannot be fulfilled, and $M_R \overset{a}{\not\sim} M_R L_0$. Starting from a rhombic lattice, it follows also that $M_D L_0 \overset{a}{\not\sim} M_D$; therefore, the lattice is of sort RD.

Proposition 6. Suppose $L \neq \pm E$ is a proper automorph of a primitive lattice Λ, and let S be an element of $GL(2, Z)$ such that:

$$S^{-1}LS = L. \tag{3.16}$$

Then S is either a proper automorph or an improper negautomorph of Λ.

Proof. Suppose that $g(B) = (a, b, c)$ is a primitive metric tensor of Λ with $a > 0$. According to the hypothesis

$$L = \begin{pmatrix} \frac{1}{2}(n - bu) & -cu \\ au & \frac{1}{2}(n + bu) \end{pmatrix} \in SL(2, Z)$$

where $u \neq 0$. Putting

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

relation (3.16) implies

$$\beta = -\frac{c}{a}\gamma \quad \text{and} \quad \delta = \alpha + \frac{b}{a}\gamma. \tag{3.17}$$

One then finds:

$$S^t g(B)\, S = (\det S)\, g(B), \tag{3.18}$$

which proves the proposition.

Proposition 7. Suppose that $L \neq \pm E$ is a proper automorph of a primitive lattice Λ, and let S be an element of $GL(2, Z)$ such that

$$S^{-1}LS = L^{-1}. \tag{3.19}$$

Then S is either an improper automorph or a proper negautomorph of Λ.

Proof. Consider $g(B)$. L and S as given above. Relation (3.19) implies

$$\alpha = -\delta \quad \text{and} \quad \beta = \frac{b}{a}\alpha + \frac{c}{a}\gamma. \tag{3.20}$$

One then finds:

$$S^t g(B)\, S = (-\det S)\, g(B). \tag{3.21}$$

We are now ready for the investigation of the crystallographic point groups.

4. *Equivalence classes of relativistic point groups.* Dealing with relativistic point groups, one naturally arrives at the conclusion that various equivalence classes between them have to be defined. This problem, however, is much more delicate than in the Euclidean case. Let us therefore indicate once again those various equivalences which we are considering (cf. section 2 of ref. 1 and refs. 5 and 7).

(i) Abstract point groups. They are the representatives of the isomorphism classes of point groups; the equivalence relation is

$$K_1 \overset{\mathrm{i}}{\sim} K_2 \qquad \text{if} \qquad K_1 \cong K_2. \tag{4.1}$$

(ii) Arithmetic point groups. They are the representatives of the conjugation classes of point groups in $GL(2, Z)$ (arithmetic crystal classes); the equivalence relation is

$$K_1 \overset{\mathrm{a}}{\sim} K_2 \qquad \text{if} \qquad S^{-1}K_1S = K_2, \qquad S \in GL(2, Z). \tag{4.2}$$

(iii) Geometric point groups. They are the representatives of the conjugation classes of point groups in $GL(2, R)$ (geometric crystal classes); the equivalence relation is

$$K_1 \overset{\mathrm{g}}{\sim} K_2 \qquad \text{if} \qquad S^{-1}K_1S = K_2, \qquad S \in GL(2, R). \tag{4.3}$$

(iv) Relativistic geometric point groups. They are the representatives of the conjugation classes of point groups in the Lorentz group $O(1,1)$ (relativistic geometric crystal classes). For deciding whether two subgroups K_1 and K_2 of $GL(2, Z)$ are conjugate subgroups of $O(1,1)$, it is necessary to indicate the bases B_1 and B_2 to which they are referred. Then $(K_1, B_1) \overset{\mathrm{rg}}{\sim} (K_2, B_2)$ if, for S_1 and $S_2 \in GL(2, R)$ such that $B = B_1S_1$ and $B = B_2S_2$ one has

$$S_2L^{-1}S_1^{-1}K_1S_1LS_2^{-1} = K_2, \qquad L \in O(1, 1). \tag{4.4}$$

The metric defined by B is $g(B) = (1, 0, -1)$. Actually, if for

$$g(B_1) = (a_1, b_1, c_1)$$

and $g(B_2) = (a_2, b_2, c_2)$, sign $a_1 =$ sign a_2, then $K_1 \overset{\mathrm{g}}{\sim} K_2$ already implies

$$(K_1, B_1) \overset{\mathrm{rg}}{\sim} (K_2, B_2).$$

(v) Relativistic arithmetic point groups. They are the representatives of the relativistic arithmetic equivalence classes (relativistic arithmetic crystal classes). For deciding whether two subgroups K_1 and

K_2 of $GL(2, Z)$ belong to one and the same relativistic arithmetic equivalence class it is necessary to indicate the basis B_1 and B_2 to which they are referred. Then $(K_1, B_1) \overset{\text{ra}}{\sim} (K_2, B_2)$ if, for $S \in GL(2, Z)$ one has $(k \in R, k > 0)$:

$$S^{-1}K_1S = K_2, \qquad S^{\text{t}}g(B_1)\,S = kg(B_2). \tag{4.5}$$

We shall not give an explicit classification of relativistic arithmetic point groups, because they simply are obtained by adjoining to each arithmetic crystal class the similarity classes of all possible bases of lattices left invariant by the point group (see for more details ref. 5, part I).

5. *Point groups of finite order.* The only automorphs of finite order are the identity E, the total inversion $-E$, and traceless improper elements M of $GL(2, Z)$. There are three finite abstract point groups given by the following generators and relations:

$$\begin{aligned} &C_1 \cong \{E\}, && \\ &C_2 \cong \{-E\} \cong \{M\}, && (-E)^2 = (M)^2 = E, \\ &D_2 \cong \{-E, M\}, && (-E)^2 = (M)^2 = (-EM)^2 = E. \end{aligned} \tag{5.1}$$

The corresponding geometric crystal classes are four:

$$K_1 = \{E\}, \qquad K_2 = \{-E\}, \qquad K_3 = \{M\} \qquad \text{and}$$
$$K_4 = \{-E, M\}; \tag{5.2}$$

they split into the following six arithmetic crystal classes:

$$\begin{aligned} &K_1\ = \{E\}, && K_2\ = \{-E\}, \\ &K_{3R} = \{M_R\}, && K_{3D} = \{M_D\}, \\ &K_{4R} = \{-E, M_R\}, && K_{4D} = \{-E, M_D\} \end{aligned} \tag{5.3}$$

where one may take as representative for the rectangular and for the rhombic class

$$M_R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_D = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

respectively.

The relativistic geometric crystal classes correspond to another splitting of (5.2) and are usually meant when one speaks of point groups in general. We therefore introduce for them (as in the Euclidean case) special symbols representing a generalization of the international notation. Clearly the only class that splits is K_3 according to whether the lattice direction left invariant by the mirror M is space-like (K_{3+}) or time-like (K_{3-}). As representatives of the relativistic geometric classes we have:

$$\left.\begin{array}{ll} K_1 = \{E\} \overset{\text{Def}}{=} 1, & \\ K_2 = \{-E\} \overset{\text{Def}}{=} \bar{1}, & \\ K_{3+} = \{M\} \overset{\text{Def}}{=} r, & M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\text{: “time-inversion”} \\ K_{3-} = \{-M\} \overset{\text{Def}}{=} m, & -M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\text{: “space-inversion”} \\ K_4 = \{-E, M\} \overset{\text{Def}}{=} mr. & \end{array}\right\} \quad (5.4)$$

In (5.4) we suppose that the first basis-vector of the basis to which these groups are referred is space-like.

6. *Proper point groups of finite order.* We first prove the following proposition on the number of generators of infinite order.

Proposition 8. In a (two-dimensional) infinite relativistic point group, the number of generators of infinite order can always be reduced to one.

Proof. Without restriction of generality we may suppose that the point group K leaves invariant a primitive metric tensor $g(B) = (a, b, c)$ with $a > 0$ and $d = b^2 - 4ac$ not a squared integer. Consider two generators, L_1 and L_2, of K of infinite order. They are necessarily of the form:

$$L_1 = \pm L_0^{k_1}, \quad L_2 = \pm L_0^{k_2}; \quad k_1, k_2 \in Z \quad (k_1, k_2 \neq 0) \quad (6.1)$$

where L_0 is the fundamental Lorentz transformation (3.2) of $g(B)$. We have to treat various cases and we do this by means of a number of lemmas.

Lemma 8.1. Put $gcd(k_1, k_2) = k$; then

$$K = \{L_0^{k_1}, L_0^{k_2}\} = \{L_0^{k}\}. \quad (6.2)$$

Proof. We may suppose $k_1, k_2 > 0$. Then there are $z_1, z_2 \in Z$ such that $k = z_1k_1 + z_2k_2$. (See ref. 8, p. 2.)

Lemma 8.2. Put $gcd(k_1, k_2) = k$, $k_1 = x_1k$ and $k_2 = x_2k$. Then if x_1 or x_2 is even:

$$K = \{-L_0^{k_1}, -L_0^{k_2}\} = \{-E, L_0^{k}\}; \quad (6.3a)$$

if x_1 and x_2 are odd:

$$K = \{-L_0^{k_1}, -L_0^{k_2}\} = \{-L_0^{k}\}. \quad (6.3b)$$

(By $-L$ we mean $-EL$.)

Proof. x_1 and x_2 are relatively prime and we may suppose without restriction x_1 odd. In this case $gcd(x_1, x_2) = 1$ and there are $z_1, z_2 \in Z$ such that $x_1z_1 + 2z_2x_2 = 1$.

One has

$$L_1^{z_1} L_2^{2z_2} = -L_0^{k_1 z_1} L_0^{2k_2 z_2} = -L_0^k$$

and $(-L_0^k)^{x_1} = -L_0^{k_1}$. If x_2 is odd $(-L_0^k)^{x_2} = -L_0^{k_2}$, whereas if it is even $(-L_0^k)^{x_2} = L_0^{k_2}$ so that $-E$ also belongs to the group K.

Lemma 8.3. Put $gcd(k_1, k_2) = k$, $k_1 = x_1 k$ and $k_2 = x_2 k$. If x_1 is even then

$$\{L_0^{k_1}, -L_0^{k_2}\} = \{-L_0^k\}; \tag{6.4}$$

if x_1 is odd, then

$$\{L_0^{k_1}, -L_0^{k_2}\} = \{-E, L_0^k\}. \tag{6.5}$$

Proof. In the first case, x_1 even, x_2 is necessarily odd, and there are z_1, $z_2 \in Z$ such that: $x_1 z_1 + x_2 z_2 = 1$. But then z_2 is odd and

$$L_1^{z_1} L_2^{z_2} = L_0^{k_1 z_1} (-L_0^{k_2})^{z_2} = -L_0^k.$$

In the second case, x_1 odd, x_1 and $2x_2$ are relatively prime, and there are z_1, $z_2 \in Z$ such that $z_1 x_1 + 2z_2 x_2 = 1$. Therefore:

$$L_1^{z_1} L_2^{2z_2} = L_0^{z_1 k_1} (-L_0^{k_2})^{2z_2} = L_0^k.$$

This means that $(L_0^k)^{-x_2} = L_0^{-k_2}$ also belongs to K and with $-L^{k_2}$, so does $-E$, which can be taken as second generator, and is of finite order. The three lemmas together prove proposition 8.

It is now easy to derive the various proper point groups of infinite order. The abstract point groups are only two:

$$C_\infty \cong \{L\},$$

$$C_\infty \times C_2 \cong \{-E, L\}, \qquad (-E)^2 = L^{-1}(-E)\, L = E. \tag{6.6}$$

The generator L is any element of $SL(2, Z)$ with $|\mathrm{tr}\, L| > 2$. There is an infinite number of corresponding crystal classes: one for each value n of the trace of L in the case of C_∞, and one for each absolute value $|n|$ of the same trace in the case of $C_\infty \times C_2$.

One has:

$$\begin{aligned} K_{5,n} &= \{L_n\}, & L_n &\in SL(2, Z), & \mathrm{tr}\, L_n &= n, \\ K_{6,|n|} &= \{-E, L_n\}, & n &\in Z, & |n| &> 2. \end{aligned} \tag{6.7}$$

To see this, consider two proper elements L_n' and L_n'' with trace n. L_n' and L_n'', or possibly $(L_n'')^{-1}$, are similar to one and the same Lorentz transformation. Furthermore, one also has:

$$\{-E, L_n\} \overset{g}{\sim} \{-E, L_{-n}\}.$$

This argument shows at the same time that (6.7) also gives the relativistic geometric classes:

$$K_{5,n} = \{L_n\} \stackrel{\text{Def}}{=} \hat{n} \qquad \text{and} \qquad K_{6,|n|} = \{-E, L_n\} \stackrel{\text{Def}}{=} \bar{1}\hat{n}. \tag{6.8}$$

In the following discussion of the arithmetic point groups, attention must be paid to the distinction between the arithmetic classes of elements of $GL(2, Z)$ and the arithmetic classes of subgroups of $GL(2, Z)$.

Consider any element L of $SL(2, Z)$ with absolute value of the trace greater than two. Then groups generated by any element belonging to the arithmetic class of L, or to that of L^{-1}, are in the arithmetic class of the group $\{L\}$ generated by L. Note that the arithmetic class of L coincides with that of L^{-1} if and only if L is an automorph of a lattice of type I, II or III (see table 6.9).

In the same way one gets all the different arithmetic point groups $\{-E, L\}$ by taking for L a representative of the union of the arithmetic classes of L, L^{-1}, $-L$ and $-L^{-1}$.

We now explain how representatives L can be found, for $|n| < 25$, using the data reported in the appendix of ref. 1. The pairs $\{g(B)\}$ and $\{-g(B)\}$ of arithmetic classes of inverse primitive quadratic form (with a discriminant that is not a squared integer) are in one-to-one correspondence with the arithmetic classes of point group $\{-E, L_0\}$, where L_0 is the fundamental Lorentz transformation of $\pm g(B)$.

Actually, in the appendix mentioned above, the Bravais lattices may have an orientation, and the quadratic forms indicated are representatives of the proper arithmetic classes. Enantiomorphic as well as inverse lattices are left invariant by the same automorphs, so that it is sufficient to take fundamental Lorentz transformation L_0 of one representative of the union of the four corresponding proper arithmetic classes of primitive quadratic form. Possibly some of these classes coincide. The general situation is presented in table (6.9).

TABLE (6.9)

Correspondence between oriented Bravais lattices and arithmetic classes of automorphs of infinite order

Lattice type	Oriented Bravais lattices	Arithmetic classes of automorphs of infinite order
I	▲ = △ = ▼ = ▽	● = ○
II	▲ = △, ▼ = ▽	● = ○
III	▲ = ▼, △ = ▽	● = ○
IV	▲ = ▽, ▼ = △	●, ○
V	▲, △, ▼, ▽	●, ○

The symbols used in table (6.9) have the following meaning:

▲: $\{g(B)\}_+$ is the proper arithmetic class of $g(B)$, a primitive (non-isotropic) metric tensor.

△: $\{g'(B)\}_+$, where $g'(B)$ is enantiomorphic to $g(B)$.

▼: $\{-g(B)\}_+$.

▽: $\{-g'(B)_+$.

●: the arithmetic class of L, a proper automorph of infinite order of $g(B)$.

○: the arithmetic class of L^{-1}.

Table (6.10) gives the generators L_0 of arithmetic point groups isomorphic either to C_∞ or to $C_\infty \times C_2$. To obtain all these point groups for given n, one has to take in the first case all groups $\{L_0\}$ and $\{-L_0\}$, whereas in the second case $\{-E, L_0\}$ already gives all arithmetic point groups of that type.

TABLE (6.10)

Representatives L_0 of generators of arithmetic point groups isomorphic either to C_∞ or to $C_\infty \times C_2$ $(2 < |\mathrm{Tr}\, L_0| < 12)$

n	u	d	$g(B)$	$\pm L_0$
3	1	5	(1, 1, −1)	$\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$
4	1	12	(1, 2, −2) (2, 2, −1)	$\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$
5	1	21	(1, 3, −3) (3, 3, −1)	$\begin{pmatrix}1 & 3\\ 1 & 4\end{pmatrix}$
6	1	32	(1, 4, −4) (4, 4, −1)	$\begin{pmatrix}1 & 4\\ 1 & 5\end{pmatrix}$
	2	8	(1, 2, −1)	$\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}$
7	1	45	(1, 5, −5) (5, 5, −1)	$\begin{pmatrix}1 & 5\\ 1 & 6\end{pmatrix}$
	3	5	(1, 1, −1)	$\begin{pmatrix}2 & 3\\ 3 & 5\end{pmatrix}$
8	1	60	(1, 6, −6) (6, 6, −1)	$\begin{pmatrix}1 & 6\\ 1 & 7\end{pmatrix}$
			(2, 6, −3) (3, 6, −2)	$\begin{pmatrix}1 & 3\\ 1 & 7\end{pmatrix}$
9	1	77	(1, 7, −7) (7, 7, −1)	$\begin{pmatrix}1 & 7\\ 1 & 8\end{pmatrix}$
10	1	96	(1, 8, −8) (8, 8, −1)	$\begin{pmatrix}1 & 8\\ 1 & 9\end{pmatrix}$
			(3, 6, −5) (5, 6, −3)	$\begin{pmatrix}2 & 5\\ 3 & 8\end{pmatrix}$
	2	24	(1, 4, −2) (2, 4, −1)	$\begin{pmatrix}1 & 4\\ 2 & 9\end{pmatrix}$
11	1	117	(1, 9, −9) (9, 9, −1)	$\begin{pmatrix}1 & 9\\ 1 & 10\end{pmatrix}$
	3	13	(1, 3, −1)	$\begin{pmatrix}1 & 3\\ 3 & 10\end{pmatrix}$
12	1	140	(1, 10, −10) (10, 10, −1)	$\begin{pmatrix}1 & 10\\ 1 & 11\end{pmatrix}$
			(2, 10, −5) (5, 10, −2)	$\begin{pmatrix}1 & 5\\ 2 & 11\end{pmatrix}$

7. *Improper point groups of infinite order.* The following proposition, together with proposition 8, limits to three the largest minimal number of generators in a (two-dimensional) relativistic point group.

Proposition 9. In a relativistic point group the number of improper generators can always be reduced to one.

Proof. Substitute the set $(M_1, M_2, \ldots, M_j)$ of improper generators by the equivalent set of generators $(M_1, M_1M_2, \ldots, M_1M_j)$.

It follows that the only abstract point groups we have not yet considered are:

$$D_\infty = \{M, L\}, \qquad M^2 = (ML)^2 = E, \tag{7.1}$$
$$D_\infty \times C_2 = \{-E, M, L\}, \qquad (-E)^2 = M^2 = (ML)^2 = L^{-1}(-E)\,L = E.$$

The corresponding geometric crystal classes are ($n \in Z$, $|n| > 2$):

$$K_{7,n} = \{M, L_n\}, \qquad K_{8,|n|} = \{-E, M, L_n\} \tag{7.2}$$

with M as in (7.1) and $L_n \in SL(2, Z)$ such that tr $L_n = n$. For checking the correctness of (7.2), consider the point groups referred to the orthonormal axis $B = (e_1, e_2)$ with $g(B) = (1,0,-1)$, and choose e_1 (or e_2) in the mirror plane of M, if M is space-like (or time-like). One then has

$$L_n = \pm\begin{pmatrix}\cosh\chi & \sinh\chi\\ \sinh\chi & \cosh\chi\end{pmatrix} = \pm L(\chi);$$
$$M = \begin{cases}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} & \text{for } M \text{ space-like,}\\ \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} & \text{for } M \text{ time-like.}\end{cases} \tag{7.3}$$

The following conjugations hold

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} L(\chi)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = L(\chi), \qquad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} M\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = -M.$$

This same reasoning shows that $K_{7,n}$ splits into two relativistic geometric classes, whereas $K_{8,|n|}$ remains one class. We thus have the following relativistic geometric crystal classes

$$\begin{aligned} K_{7^+,n} &= \{M, L_n\} = \hat{n}r,\\ K_{7^-,n} &= \{-M, L_n\} = \hat{n}m,\\ K_{8,|n|} &= \{-E, M, L_n\} = \hat{n}mr, \end{aligned} \tag{7.4}$$

where M is a space-like mirror.

To discuss the arithmetic crystal classes, we need to take into account whether lattices left invariant are of type I or of type II (see section 3, and ref. 1, section 4).

(i) Type I lattices (ambiguous, stable and negative). We discuss first the arithmetic equivalence of two point groups, K and K_1', given by:

$$K = \{M, L\}, \qquad K_1' = \{M_1', L_1'\} \tag{7.5}$$

with M and M_1' improper automorphs, and L, L_1' proper automorphs of infinite order of type I lattices. Suppose $K \overset{a}{\sim} K_1'$; then there is $S_1 \in GL(2, Z)$ such that:

$$S_1^{-1}L_1'S_1 = L \qquad \text{or} \qquad S_1^{-1}L_1'S_1 = L^{-1} \tag{7.6}$$

because $L^{\pm 1}$ and $L_1'^{\pm 1}$ are the only generators of infinite order. This means that we may restrict the discussion to point groups given by

$$K = \{M, L\}, \qquad K_1 = \{M_1, L\} \tag{7.7}$$

where

$$L = \pm L_0^k M, \qquad M_1 = \pm ML_0^j; \qquad k, j \in Z, \qquad k \neq 0 \tag{7.8}$$

and where L_0 is the fundamental Lorentz transformation of the lattices left invariant.

Arithmetic equivalence between K and K_1 implies that there is $S \in GL(2, Z)$ with:

$$\text{(a)}\ \ L = S^{-1}LS \qquad \text{or} \qquad \text{(b)}\ \ L^{-1} = S^{-1}LS \tag{7.9}$$

and

$$M = S^{-1}M_1L^zS, \qquad \forall z \in Z. \tag{7.10}$$

In the case (7.9)(a), we know from proposition 6 that S is either a proper automorph or an improper negautomorph, therefore of the form

$$S = \pm A_0^{-s}, \qquad (\forall s \in Z) \tag{7.11}$$

where A_0 is the fundamental improper negautomorph of the type I lattice considered. Using $A_0^2 = L_0$, and the conjugation properties indicated in table (3.5) of ref. 1, one obtains the condition:

$$M = A_0^s M_1 L^z A_0^{-s} = (-1)^s M_1 L^z L_0^{-s}. \tag{7.12}$$

In the case (7.9)(b), one applies proposition 7, and S can be written as

$$S = \pm A_0^{-s} M, \tag{7.13}$$

giving exactly condition (7.12). We have to consider several cases:

$$\text{1st case:} \qquad L = L_0^k, \qquad M_1 = ML_0^j; \qquad k, j \in Z, \qquad k \neq 0. \tag{7.14}$$

Relation (7.12) yields

$$M = (-1)^s\, ML_0^{j+kz-s},$$

that is to say $s = 2t$ and $s = j + kz$.

Arithmetic equivalence then corresponds to the existence of solutions of the congruence:

$$2t \equiv j \pmod{k}. \tag{7.15}$$

For odd k there is a solution for all $j \in Z$; but for even k there is a solution only for all even j.

2nd case: $L = L_0^k, \qquad M_1 = -ML_0^j.$ (7.16)

The condition for arithmetic equivalence is now expressed by:

$$2t \equiv (j - 1) \bmod k. \tag{7.17}$$

3rd case: $L = -L_0^k, \qquad M_1 = ML_0^j.$ (7.18)

One now obtains

$$2t \equiv j \bmod (k - 1). \tag{7.19}$$

4th case: $L = -L_0^k, \qquad M_1 = -ML_0^j$ (7.20)

give

$$2t \equiv (j - 1) \bmod (k - 1). \tag{7.21}$$

Considering now all four possible cases, and corresponding equivalences, one obtains the following arithmetic classes ($\forall k \in Z,\ k > 0$):

$$\begin{matrix} \{M, L_0^{2k+1}\}, & \{M, L_0^{2k}\}, & \{ML_0, L_0^{2k}\}, \\ \{M, -L_0^{2k}\}, & \{M, -L_0^{2k-1}\}, & \{ML_0, -L_0^{2k-1}\}. \end{matrix} \tag{7.22}$$

Here L_0 is the fundamental Lorentz transformation of a representative $g(B)$ of an arithmetic class of primitive lattices of type I. For each such L_0 there are the arithmetic classes (7.22). For M, one may always take M_R if the lattice is rectangular and M_D if it is rhombic.

Going over now to the arithmetic equivalence of the groups

$$K = \{-E, M, L\}, \qquad K_1 = \{-E, M_1, L\}, \tag{7.23}$$

one sees that the previous four cases (7.14), (7.16), (7.18) and (7.20) all occur, so that for the arithmetic equivalence between K and K_1 with M and L as in (7.8), it is sufficient that there be a solution of one of the four congruences (7.15), (7.17), (7.19) or (7.21). Clearly, therefore, there are the arithmetic classes given by:

$$\{-E, M, L_0^k\}, \qquad \forall k \in Z, \qquad k \neq 0 \tag{7.24}$$

with L_0 and M as above.

(ii) Type II lattices (ambiguous, unstable and positive). Relations (7.5) to (7.10) can simply be taken over from the previous discussion. The difference now is that no negautomorph is admitted. Therefore in the case (7.9)(a), it follows from proposition 6 that S is necessarily of the form:

$$S = \pm L_0^{-t} \qquad (\forall t \in Z) \tag{7.25}$$

giving the condition of arithmetic equivalence between K and K_1 of (7.7) as:

$$M = M_1 L^z L_0^{-2t}. \tag{7.26}$$

Relation (7.9)(b) yields again the same condition, and can therefore be disregarded. Here also we have to consider four cases.

The congruence obtained in the case (7.14) is again (7.15). In the second case, (7.16), the equivalence condition

$$(-1)\, L_0^{j+kz-2t} = 1 \tag{7.27}$$

is never fulfilled. Case (7.18) gives

$$2t \equiv j \pmod{2k} \tag{7.28}$$

and case (7.20)

$$2t = j + kz, \qquad \text{odd } z. \tag{7.29}$$

Relation (7.29) always has solutions if k and j are equal modulo two, otherwise never. Altogether one arrives at the following arithmetic classes ($\forall k \in Z,\, k > 0$):

$$\begin{array}{lll} \{M, L_0^{2k-1}\}, & \{M, L_0^{2k}\}, & \{ML_0, L_0^{2k}\}, \\ \{-M, L_0^{2k-1}\}, & \{-M, L_0^{2k}\}, & \{-ML_0, L_0^{2k}\}. \\ \{M, -L_0^{k}\}, & \{ML_0, -L_0^{k}\}, & \end{array} \tag{7.30}$$

Here L_0 is the fundamental Lorentz transformation of an arithmetic class of either a primitive lattice of type II or of its inverse lattice. For M, one takes the mirror M_R if the lattice is IIR, and M_D if it is II D, one takes either mirror in the mixed case II RD.

The point groups (7.23) are equivalent if at least one of the congruences (7.15), (7.28) and (7.29) is satisfied. One sees that for odd k, there is always a solution, whereas for even k, j has to be even too for a solution to exist. There are therefore the following arithmetic crystal classes:

$$\{-E, M, L_0^{2k+1}\}, \qquad \{-E, M, L_0^{2k}\}, \qquad \{-E, ML_0, L_0^{2k}\} \tag{7.31}$$

with M and L_0 as in the case (7.30).

8. *Remarks on the Euclidean case.* The methods discussed in this paper are essentially valid also in the Euclidean case. Of course, a number of classes which are distinct in the indefinite case now coincide (owing to the

fact that the fundamental automorph is of finite order). Let us illustrate the improper case discussed in section 7 for:

$$L_0 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \tag{8.1}$$

a proper fundamental automorph of the primitive lattice of type II D corresponding to the metric tensor $g(B) = (1, 1, 1)$ of discriminant $d = -3$. One has

$$L_0^6 = E, \qquad L_0^3 = -E. \tag{8.2}$$

There are essentially only two different proper generators of order different from two: L_0 and L_0^2; in this particular case, the classes indicated in (7.30) give rise to the following arithmetic classes:

$$\{M, L_0\} = \{M, -L_0^2\} \sim \{-M, L_0\} = \{M, L_0^3, L_0\},$$
$$\{M, L_0^2\} = \{M, -L_0\} \sim \{-ML_0, L_0^2\},$$
$$\{ML_0, L_0^2\} = \{ML_0, -L_0\} \sim \{-M, L_0^2\}.$$

With $M = M_D$, the arithmetic point groups are

$$\{M_D, L_0\} = \text{p6m}, \qquad \{M_D, L_0^2\} = \text{p31}m, \qquad \{M_DL_0, L_0^2\} = \text{p3}m\text{1}$$

where the notation of the symorphic space groups is used for labelling the arithmetic classes.

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