$$R \approx R_\infty r/\sqrt{a^2+r^2} \qquad (6)$$

where R_∞ is a normalization constant and "a" is the radius of the vortex "core" which is given, in terms of an effective mass $m^*(=m[1-W_2\sigma_n 2m/\hbar^2]^{-1}$ by

$$a = \sqrt{\hbar^2/2m^* R_\infty^2 W_1 + W_2/W_1} \; . \qquad (7)$$

For $W_2 = 0$ (as is the case when the interatomic potential is taken to be δ-function like) $m^* = m$ and Fetter's result - $a = \sqrt{\hbar^2/2mR_\infty^2 W_1}$ - is retrieved; in this case, however, $\omega_c = \infty$. For $W_2 \neq 0$ the mass renormalisation persists even at 0^oK since here σ_n is still finite*.

It is to be concluded, therefore, that the interaction between the superfluid and normal components of liquid ^{4}He II which is contained in Fröhlich's macroscopic wave equation is capable of generating ** quantized vortices not only of the usual type - existing for general ω_0 [3,4] - but also one which exists for only one frequency ω_c, exhibiting only unit quantum of circulation.

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** It should be realized that existing treatments (refs. 3 and 4), by their restriction to a δ-function model for the interatomic ^{4}He potential, deny themselves the possibility of such a generative coupling even at non zero temperatures where σ_n is finite; for the coupling derived from W_2 which in these treatments vanishes.

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RELATIVISTIC SYMMETRY GROUPS OF UNIFORM ELECTROMAGNETIC FIELDS

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The relativistic symmetry groups for all possible uniform electromagnetic fields are presented and discussed.

Relativistic symmetry groups of uniform electromagnetic fields (i.e., those which are space- and time-independent) are the semi-direct product of the group of all translations in space and time by the largest subgroup K of the Lorentz group leaving the corresponding field invariant (the subgroup K is called relativistic point group). They are thus symmorphic subgroups of the Poincaré group.

Uniform electromagnetic fields occur in one of the following five cases: (i) magnetic field only (H): $\boldsymbol{H} \neq 0$, $\boldsymbol{E} = 0$; (ii) electric field only (E): $\boldsymbol{H} = 0$, $\boldsymbol{E} \neq 0$; (iii) parallel fields (//): $\boldsymbol{H} /\!/ \boldsymbol{E}$, $\boldsymbol{H} \neq 0$, $\boldsymbol{E} \neq 0$; (iv) perpendicular fields (⊥): $\boldsymbol{H} \perp \boldsymbol{E}$, $\boldsymbol{H} \neq 0$, $\boldsymbol{E} \neq 0$; (v) oblique fields (V): $\boldsymbol{H}\cdot\boldsymbol{E} \neq 0$, $\boldsymbol{H}\cdot\boldsymbol{E} \neq |\boldsymbol{H}|\cdot|\boldsymbol{E}|$.

The relativistic point group of any of these fields is larger than its Shubnikov group usually indicated in the literature. Thus for a magnetic field $\boldsymbol{H}$ in the z-direction, the Shubnikov point group P_H (in the international notation and using a prime to denote time inversion) is $\frac{\infty}{m}\frac{2'}{m'}\frac{2'}{m'}$ [e.g., 1], whereas its relativistic point group K_H is:

$$K_H = \{m'_x,\ \bar{1},\ R_z(\phi),\ L_z(\chi) | \forall \phi,\ \chi \in R\}. \quad (1)$$

In the brackets a set of generators is given: m'_x is a mirror perpendicular to the x-axis (m_x) combined with time inversion (1'); $\bar{1}$ is the space inversion; $R_z(\phi)$ is any rotation of angle around the z-axis, $L_z(\chi)$ any special Lorentz transformation with velocity βc (i.e., between two inertial systems only differing in their relative velocity βc) in the z-direction and $\cosh\chi = 1/\sqrt{1-\beta^2}$. For an electric field $\boldsymbol{E}$ along the z-axis, one finds:

$$K_E = \{1',\ m_x,\ R_z(\phi),\ L_z(\chi) | \forall \phi,\ \chi \in R\}. \quad (2)$$

The relativistic point group for the case of parallel fields is simply:

$$K_{\parallel} = \{m'_x,\ R_z(\phi),\ L_z(\chi) | \forall \phi,\ \chi \in R\} = K_H \cap K_E \quad (3)$$

all the remaining cases we choose the z-axis the direction of $\boldsymbol{E}$ and the x-axis in that of $\boldsymbol{H}_\perp$ he component of $\boldsymbol{H}$ perpendicular to $\boldsymbol{E}$).

When the fields are perpendicular, one has to distinguish between the three cases: $|\boldsymbol{H}|$ greater, equal, smaller than $|\boldsymbol{E}|$. (We use here Gauss units.) Defining $a = |\boldsymbol{H}|/|\boldsymbol{E}|$, one has for $a^2 = 1$:

$$K_\perp(a^2=1) = \{m'_y,\ m_x,\ L(\sigma),\ \bar{L}(\rho) | \forall \sigma,\ \rho \in R\}, \quad (4)$$

where

$$L(\sigma) = \begin{pmatrix} 1+\frac{1}{2}\sigma^2 & \sigma & -\frac{1}{2}\sigma^2 & 0 \\ \sigma & 1 & -\sigma & 0 \\ \frac{1}{2}\sigma^2 & \sigma & 1-\frac{1}{2}\sigma^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\bar{L}(\rho) = \begin{pmatrix} 1+\frac{1}{2}\rho^2 & 0 & -\frac{1}{2}\rho^2 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2}\rho^2 & 0 & 1-\frac{1}{2}\rho^2 & \rho \\ \rho & 0 & -\rho & 1 \end{pmatrix}$$

For $a^2 < 1$, the Lorentz transformation $S(a)$ with velocity ac in the y-direction transforms the magnetic field to zero, so that:

$$K_\perp(a) = S^{-1}(a)\, K_E\, S(a), \qquad a^2 < 1. \quad (5)$$

For $a^2 > 1$, it is the electric field that is transformed to zero by the Lorentz transformation $\bar{S}(a)$ with velocity c/a along the y-axis, so that:

$$K_\perp(a) = \bar{S}^{-1}(a)\, K_H\, \bar{S}(a), \qquad a^2 > 1. \quad (6)$$

In the case of oblique fields, one defines $|\boldsymbol{E}| = E$, $|\boldsymbol{H}_\perp| = aE$ and $|\boldsymbol{H}_\parallel| = bE$. The Lorentz transformation $S(a,b)$ then transforms this case to that of parallel fields; the magnetic field $\boldsymbol{q}$ and electric field $\boldsymbol{p}$ in the z-direction are given by:

$$|\boldsymbol{p}| \overset{\text{def.}}{=} p = \frac{1}{\sqrt{2}}\left[-a^2-b^2+1+\sqrt{(a^2+b^2-1)^2+4b^2}\right]^{\frac{1}{2}}$$

$$|\boldsymbol{q}| \overset{\text{def.}}{=} q = \frac{1}{\sqrt{2}}\left[a^2+b^2-1+\sqrt{(a^2+b^2-1)^2+4b^2}\right]^{\frac{1}{2}}.$$

Expressed in these variables, $S(a,b)$ is a Lorentz transformation with velocity $c\sqrt{1-p^2}/\sqrt{1+q^2}$ along the y-axis followed by a rotation of angle $\omega = \arccos(p\sqrt{1+q^2}/\sqrt{p^2+q^2})$, around the same axis.

Thus

$$K_{/}(a,b) = S^{-1}(a,b)\, K_\parallel\, S(a,b). \quad (7)$$

A more detailed account of the present work will be published elsewhere.

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