Space-Time Symmetry of Linearly Polarized Electromagnetic Plane Waves.

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The symmetry in space and time of a transverse electromagnetic (TEM) plane wave with (isotropic) four-vector k is the largest subgroup G^k of the Poincaré group $IO_{3,1}$ leaving the electromagnetic-field tensor invariant.

The subgroup T^k of G^k, consisting only of translations in space and time, is a normal subgroup of G^k. The elements of T^k are called primitive translations. The factor group G^k/T^k is isomorphic to a subgroup K^k of the Lorentz group. K^k is the point group of the electromagnetic wave in question.

As the Poincaré group is the semi-direct product of the group T of all translations in space and time and the Lorentz group $O_{3,1}$, we write the elements of $G^k \subset IO_{3,1}$ as (t, L) with $t \in T$ and $L \in K^k \subset O_{3,1}$. Their multiplication rule is given by

$$(t_1, L_1)(t_2, L_2) = (t_1 + L_1 t_2, L_1 L_2) .$$

If $t \notin T^k$, then t can be written as

$$t = a + u(L) \tag{1}$$

with $a \in T^k$.

The element $u(L)$ is a nonprimitive translation associated to L. One has [1]

$$u(L_1 L_2) = u(L_1) + L_1 u(L_2) , \tag{2}$$

so that it is sufficient to know the nonprimitive transitions associated to the generators of K^k. In the case of a linearly polarized TEM wave, an orthonormal

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[1] E. ASCHER and A. JANNER: *Helv. Phys. Acta*, 38, 551 (1965); *Comm. Math. Phys.*, 11, 138 (1968).

basis e_α ($\alpha = 0, 1, 2, 3$) can be chosen in the Minkowski space with metric tensor $g_{\alpha\beta} = e_\alpha \cdot e_\beta$ (where $-g_{00} = g_{11} = g_{22} = g_{33} = 1$ and $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$) in such a way that the electromagnetic-field tensor is

$$F^{\alpha\beta}(x) = F^{\alpha\beta}(0) \cos kx \,. \tag{3}$$

Here $k^0 = k^2 = \omega/c = 2\pi/\lambda$, $k^1 = k^3 = 0$ and in Gaussian units

$$F^{\alpha\beta}(0) = E \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}. \tag{4}$$

The translation group T^k is given by

$$T^k = \{a = \mu e_1 + \nu e_3 + \varrho k + z\lambda e_2 | \forall\, \mu, \nu, \varrho \in R \text{ and } z \in Z\} \simeq R^3 \oplus Z \tag{5}$$

and the point group K^k is generated by

$$K^k = \{\bar{1}', m_x, m_y', L(\sigma), \bar{L}(\varrho) | \forall\, \sigma, \varrho \in R\}\,, \tag{6}$$

where m_x is the mirror perpendicular to the x-axis (along e_1), m_y' the mirror perpendicular to the y-axis (along e_2) followed by a time inversion, $L(\sigma)$ and $\bar{L}(\varrho)$ are Lorentz transformations belonging to the little group of the four-vector k [2]:

$$L(\sigma) = \begin{pmatrix} 1 + \frac{1}{2}\sigma^2 & \sigma & -\frac{1}{2}\sigma^2 & 0 \\ \sigma & 1 & -\sigma & 0 \\ \frac{1}{2}\sigma^2 & \sigma & 1 - \frac{1}{2}\sigma^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{L}(\varrho) = \begin{pmatrix} 1 + \frac{1}{2}\varrho^2 & 0 & -\frac{1}{2}\varrho^2 & \varrho \\ 0 & 1 & 0 & 0 \\ \frac{1}{2}\varrho^2 & 0 & 1 - \frac{1}{2}\varrho^2 & \varrho \\ \varrho & 0 & -\varrho & 1 \end{pmatrix}$$

and $\bar{1}'$ is that total (space-time) inversion.

With the choice of the origin made in (3), the only nonprimitive translation, associated to the above generators of K^k, which is nonequivalent to zero, is $u(\bar{1}')$. One may choose

$$u(\bar{1}') = \tfrac{1}{2}\lambda e_2$$

and

$$u(m_x) = u(m_y') = u(L(\sigma)) = u(\bar{L}(\varrho)) = 0\,. \tag{7}$$

Therefore G^k is a nonsymmorphic symmetry group. This means that the point group K^k is not a subgroup of G^k, or, in other words, that G^k is a nonsplit extension of T^k by K^k [1].

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[2] M. HAMERMESH: *Group Theory* (London, 1962), p. 494.

In the limit of $\lambda \to \infty$, *i.e.* of $k \to 0$, $F^{\alpha\beta}(x)$ becomes the constant $F^{\alpha\beta}(0)$, whose symmetry has already been determined ([3]). Comparison with $K_\perp$ (see (4) of ref. ([3])) shows that the only elements of $K^{k\to 0}$ not belonging to $K_\perp$ are those whose nonprimitive translations are inequivalent to zero.

A more detailed account of the present work together with the corresponding results for the cases of a circularly and an elliptically polarized TEM wave will be published elsewhere.

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([3]) A. Janner and E. Ascher: to be published.