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CRYSTALLOGRAPHIC CONCEPTS FOR INHOMOGENEOUS SUBGROUPS OF THE POINCARÉ GROUP

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Synopsis

Concepts typical for crystallographic space groups, like the group of primitive translations (U), the point group (K) and the system of non-primitive translations (u), are also very convenient in the more general case of arbitrary inhomogeneous subgroups G of the Poincaré group $IO(3, 1)$. Groups with given U and K are considered. The rôle of the cohomology groups $H^1(K, R^4/U)$ –where R^4 is the group of all space-time translations – and $H^2(K, U)$ is discussed. $H^1(K, R^4/U)$ appears if one considers the imbedding of G into $IO(3, 1)$, whereas $H^2(K, U)$ occurs if one looks at G as extension of U by K. Not every such extension gives, in general, a subgroup of $IO(3, 1)$. The elements of $H^1(K, R^4/U)$ are in one-to-one correspondence with the classes of subgroups G having given U and K, and only differing in their origins. If $H^1(K, R^4) = 0$ and if U generates the real vector space R^4, Bieberbach's conjugation theorem holds, *i.e.* abstract isomorphisms can be realized as conjugations in the affine group $A(4)$. The important consequences of this property are considered and a number of basic theorems proved. Several physical systems having inhomogeneous subgroups of the Poincaré group as symmetry groups are indicated and discussed.

1. *Introduction.* The aim of this paper is to show how concepts defined for crystallographic groups can naturally be extended and used for describing arbitrary inhomogeneous linear groups. This generalization is straightforward in the euclidean case, is possible in the affine one, but will be treated here in the relativistic (or minkowskian) case only. The reason is that treating in such a way the inhomogeneous subgroups of the Poincaré group is interesting from the physical point of view and not quite trivial mathematically.

The structure of the relativistic crystallographic groups is still under investigation. Even in the two-dimensional case (one space- and one time-dimension) only the homogeneous crystallographic groups have been derived[1]). In the four-dimensional case the classification is so far limited to those groups which are isomorphic to euclidean crystallographic groups[2]). A lot of information is of course available in current mathematical literature in a more or less direct way and may be considered to be equivalent with a knowledge of a number of relativistic crystallographic groups (see, *e.g.*, refs. 3 and 4).

However, the proofs of the fundamental theorems of euclidean crystallography – first established by Bieberbach in two famous papers[5]) and later investigated further by a number of authors[6–12]) – are based on properties that in the case of an indefinite metric are, in general, no longer true. Let us only mention, as example, the so-called imbedding theorem for abstract space groups. According to this theorem, an abstractly defined crystallographic group can always be imbedded in the group of rigid motions of a euclidean space. In this paper a relativistic imbedding theorem is given for a larger class of groups than the crystallographic ones.

Our interest for this larger class of subgroups of the Poincaré group has its first justification in physics. A first research on physical systems having symmetries in space–time and not in the 3-dimensional space only [consider *e.g.* an electromagnetic plane wave[13])] showed clearly that the frame of crystallography (even if much richer in the relativistic than in the euclidean case) is a too limited one. It became also clear that some crystallographic concepts could still be used in the new symmetry groups encountered. The presence of translations is in fact sufficient for ensuring a meaningful generalization of the crystallographic definitions. This is discussed in section 2. Section 3 is devoted to the formulation of a number of basic theorems. In section 4 the rôle of $H^1(K, R^4)$ and of $H^2(K, R^4)$ is discussed, and in the last section some examples are given. The application of symmetry groups of the type considered here to problems of relativistic physics will be the subject matter of forthcoming papers.

2. *Subgroups of the Poincaré group.* Consider the real vector space R^4 with the diagonal metric tensor g given by $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$. We define the Lorentz group $O(3,1)$ as group of real matrices

$$O(3,1) = \{\alpha \in GL(4, R) \mid \alpha^{\mathrm{T}} g \alpha = g\},$$

where α^{T} is the transpose of the matrix α. The inhomogeneous Lorentz group (or Poincaré group) $IO(3,1)$ is the semidirect product of the abelian group of translations R^4 by the group $O(3,1)$. This latter group operates, by definition, faithfully on R^4; therefore R^4 is a maximal abelian subgroup of $IO(3,1)$ (Proposition 1, ref. 11). We thus have the extension

$$0 \to R^4 \underset{(\kappa)}{\to} IO(3,1) \underset{(\sigma)}{\to} O(3,1) \to 1. \tag{2.1}$$

The monomorphism $\bar{\kappa}$ placed between parentheses denotes the injection of a subgroup. Such monomorphisms will usually be omitted in formulas. The epimorphism $\bar{\sigma}$ is the canonical projection of each element onto the coset modulo R^4 to which it belongs. After the choice of a monomorphic section $\bar{r}$:

$$\bar{r}: O(3,1) \to IO(3,1), \qquad \bar{\sigma}\bar{r} = 1_{O(3,1)} \tag{2.2}$$

the elements of $IO(3,1)$ may be written in a unique way as

$$(t, \xi) = t \cdot \bar{r}\xi \qquad t \in R^4,\ \xi \in O(3,1). \tag{2.3}$$

Take any subgroup G of $IO(3,1)$ and define the *group of primitive translations U* by

$$U = \{a \in R^4 \,|\, (a, \varepsilon) \in G\} = R^4 \cap G, \tag{2.4}$$

where ε is the unit element of $O(3,1)$. Define furthermore the *point group* K as the following subgroup of $O(3,1)$:

$$K = \{\alpha \in O(3,1) \,|\, (t, \alpha) \in G\} = G/U. \tag{2.5}$$

Property 1. The group U is free abelian and normal in G.
The group is free abelian because it is a subgroup of R^4. It is normal in consequence of its definition (2.4) and because R^4 is normal in $IO(3,1)$.

Property 2. The group G is an extension of U by K and there exists the following commutative diagram (morphism of extensions):

$$\begin{array}{ccccccccc} 0 & \to & U & \overset{(\kappa)}{\to} & G & \overset{\sigma}{\to} & K & \to & 1 \\ & & \downarrow{\scriptstyle(\iota)} & & \downarrow{\scriptstyle(\mu)} & & \downarrow{\scriptstyle(\nu)} & & \\ 0 & \to & R^4 & \underset{(\bar{\kappa})}{\to} & IO(3,1) & \underset{\bar{\sigma}}{\to} & O(3,1) & \to & 1 \end{array} \tag{2.6}$$

This follows from property 1 and the definitions of U and K. The epimorphism σ is the restriction of $\bar{\sigma}$ to G.

Here is the place to recall several relations that were used (and partly proven) in refs. 11 and 12, and that will be used extensively in this work. More about the cohomology of groups may be found in the book by MacLane[14]). (See also ref. 15.)

Let K be a group and consider a short exact sequence of K-modules

$$0 \to A \overset{\iota}{\to} B \overset{\pi}{\to} C \to 0. \tag{2.7}$$

The monomorphism ι induces monomorphisms of cochains ι_*:

$$C^q(K, A) \to C^q(K, B)$$

and homomorphisms of cohomology groups $[\iota_*]: H^q(K, A) \to H^q(K, B)$. The epimorphisms π_* and the homomorphisms $[\pi_*]$ have analogous meanings.

To (2.7) corresponds an exact sequence of cohomology groups

$$\xrightarrow{\partial_*} H^q(K, A) \xrightarrow{[\iota_*]} H^q(K, B) \xrightarrow{[\pi_*]} H^q(K, C) \xrightarrow{\partial_*} H^{q+1}(K, A) \xrightarrow{[\iota_*]} \dots \tag{2.8}$$

The sequence starts on the left with $0 \to H^0(K, A) \to$. The connecting homomorphism ∂_* is defined in the following way: the relations

$$\partial_*[v] = [m], \qquad [v] \in H^q(K, C), \qquad [m] \in H^{q+1}(K, A) \tag{2.9}$$

mean that there exist a cochain $u \in C^q(K, B)$ and cocycles $v \in Z^q(K, C)$, $m \in Z^{q+1}(K, A)$, such that

$$\iota_* m = \delta u, \qquad \pi_* u = v; \tag{2.10}$$

the homomorphism ∂_* does not depend on particular choices of u, v and m. The mapping δ is the coboundary homomorphism $\delta\colon C^q(K, B) \to C^{q+1}(K, B)$. Here we shall be concerned with the case $q = 1$. Then

$$(\delta u)(\alpha, \beta) = u(\alpha) + \alpha u(\beta) - u(\alpha\beta), \qquad \alpha, \beta \in K. \tag{2.11}$$

If now $\omega\colon K \to \bar{K}$ is a group homomorphism and A a $\bar{K}$-module, then ω induces a homomorphism of cochains $\omega^*\colon C^q(\bar{K}, A) \to C^q(K, A)$ and a homomorphism of cohomology groups $[\omega^*]\colon H^q(\bar{K}, A) \to H^q(K, A)$.

Finally, let G be an extension of an abelian group A by a group K (determining a homomorphism $\varphi\colon K \to \operatorname{Aut} A$ and a cohomology class

$$[m] \in H^2_\varphi(K, A))$$

and let $\bar{G}$ be an extension of an abelian group $\bar{A}$ by a group $\bar{K}$ (determining a homomorphism $\bar{\varphi}\colon \bar{K} \to \operatorname{Aut} \bar{A}$ and a cohomology class $[\bar{m}] \in H^2_{\bar{\varphi}}(\bar{K}, \bar{A})$). Then the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & G & \to & K & \to & 1 \qquad \varphi, [m] \\ & & \lambda\downarrow & & & & \downarrow\nu & & \\ 0 & \to & \bar{A} & \to & \bar{G} & \to & \bar{K} & \to & 1 \qquad \bar{\varphi}, [\bar{m}] \end{array} \tag{2.12}$$

can be completed by a homomorphism $\mu\colon G \to \bar{G}$ to form a morphism of extensions if an only if

$$\lambda[(\varphi\alpha)\, a] = (\bar{\varphi}\nu\alpha)(\lambda a) \tag{2.13}$$

and

$$[\lambda_*][m] = [\nu^*][\bar{m}] \in H^2_{\bar{\varphi}\nu}(K, \bar{A}). \tag{2.14}$$

Given sections $r\colon K \rightarrow\!\!- G$ and $\bar{r}\colon \bar{K} \rightarrow\!\!- \bar{G}$, the elements of G and $\bar{G}$ can be written

$$g = \langle a, \alpha\rangle, \qquad a \in A, \qquad \alpha \in K$$

and

$$\bar{g} = (\bar{a}, \bar{\alpha}), \qquad \bar{a} \in \bar{A}, \qquad \bar{\alpha} \in \bar{K}$$

respectively. Then

$$\mu\langle a, \alpha\rangle = (\lambda a + u(\alpha), \nu\alpha) \tag{2.15}$$

and there is a one-to-one correspondence between homomorphisms μ and cochains $u \in C^1_{\bar{\varphi}\nu}(K, \bar{A})$.

Note, however, that it is always possible to choose the section $\bar{r}$ so as to obtain $uK = \varepsilon \in \bar{K}$. Then, instead of (2.14), the stronger relation

$$\lambda_* m = \nu^* \bar{m} \in Z^1_{\bar{\varphi}\nu}(K, \bar{A}) \tag{2.16}$$

holds.

Let us return to the diagram (2.6) and denote the element of G as $\langle a, \alpha\rangle$. Let $[m] \in H^2(K, U)$ denote the cohomology class determined by G. After the choice of a monomorphic section $\bar{r}$, the elements of $IO(3,1)$ are denoted as (t, ξ).

Property 3.

$$\mu\langle a, \alpha\rangle = (a + u(\alpha), \alpha), \tag{2.17}$$

$$[\iota_*][m] = 0 \in H^2(K, R^4). \tag{2.18}$$

Relation (2.17) is a transcription of (2.15), relation (2.18) follows from (2.14) and the fact that $IO(3,1)$ is a semidirect product hence $[\bar{m}] = 0$.

The mapping $u: K \to R^4$ is called a *system of non-primitive translations* of the group G.

The exact sequence (2.8) allows a relationship to be established between the elements $[m]$ and u of (2.17) and (2.18).

Property 4.

$$[m] = \partial_*[\pi_* u] \in H^2(K, U), \tag{2.19}$$

where

$$\pi_* u \in Z^1(K, R^4/U). \tag{2.20}$$

From (2.18) follows

$$[m] \in \operatorname{Ker}[\iota_*] = \operatorname{Im} \partial_*.$$

Thus there exists an element $[v] \in H^1(K, R^4/U)$ such that

$$[m] = \partial_*[v].$$

By the definition of ∂_* this means that there is a nonzero $u \in C^1(K, R^4)$ and a $v \in Z^1(K, R^4/U)$ such that

$$\iota_* m = \delta u, \tag{2.21}$$

$$\pi_* u = v. \tag{2.22}$$

However, $[v]$ does not depend on particular choices of elements u, v and $m \in Z^2(K, U)$ fulfilling the above two relations. Let us recall that δ in (2.21) is the boundary homomorphism $\delta\colon C^1(K, R^4) \to C^2(K, R^4)$ defined by (2.11). Thus

$$u(\alpha\beta) \equiv u(\alpha) + \alpha u(\beta) \qquad (\text{mod } U). \tag{2.23}$$

This is exactly the relation that one obtains also by applying the definition (2.17) of u to the product $\langle 0, \alpha\rangle \langle 0, \beta\rangle$.

We can formulate also the following property:

Property 5. The element $u \in C^1(K, R^4)$ is a system of non-primitive translations of the group G in diagram (2.6) if and only if $\pi_* u \in Z^1(K, R^4/U)$.

Now we want to know how $[\pi_* u] \in H^1(K, R^4/U)$ depends on the choice of the origin of G. A change of origin of G is simply a conjugation in $IO(3, 1)$ by an element of R^4. We call *equivalent* all *systems of non-primitive translations* that differ only by a choice of origin. By a change of origin a system of non-primitive translations u is changed into the system $u + \delta d$ with $d \in R^4$ and thus $\delta d \in B^1(K, R^4)$:

$$(d, \varepsilon)(u(\alpha), \alpha)(d, \varepsilon)^{-1} = (u(\alpha) + d - \alpha d, \alpha) = (u(\alpha) + (\delta d)(\alpha), \alpha). \tag{2.24}$$

A further question is that of the dependence of a system of non-primitive translations $u \in C^1(K, R^4)$ on the choice of the monomorphic section $\hat{r}$: $O(3, 1) \to IO(3, 1)$. Since $H^1(O(3, 1), R^4) = 0$, as will be recalled in proposition 3, each change of monomorphic section corresponds to a change of origin. Therefore systems that differ by the choice of the monomorphic section are equivalent.

Property 6. Inequivalent systems of non-primitive translations for groups G as in (2.6) are in one-to-one correspondence with elements $[v]$ of the first cohomology group $H^1(K, R^4/U)$.

Let $u + \delta d$ be the system of non-primitive translations with respect to a new origin. Since π_* maps $B^1(K, R^4)$ onto $B^1(K, R^4/U)$ we have

$$[\pi_*(u + \delta d)] = [\pi_* u]. \tag{2.25}$$

Note that the classification of subgroups G of $IO(3, 1)$ according to their equivalence class $[\pi_* u]$ of non-primitive translations is a very fine one. According to (2.19), equivalent systems of non-primitive translations give rise to equivalent extensions. But inequivalent systems $[\pi_* u]$ and $[\pi_* \bar{u}]$ of non-primitive translations may be associated to a group G characterized by a given element $m \in Z^2(K, U)$ (called factor set). By (2.21) we have

$$\iota_* m = \delta u = \delta \bar{u} \tag{2.26}$$

and therefore

$$\bar{u} - u = y \in Z^1(K, R^4). \tag{2.27}$$

In general y is not a coboundary and thus $[\pi_* \bar{u}] \neq [\pi_* u]$.

It may also be seen, from (2.19), that the equivalence class of an extension G – characterized by a $[m] \in H^2(K, U)$ – may give rise to systems u and $\bar{u}$ of non-primitive translations that differ in the following way:

$$\bar{u} - u = y + \delta d + \iota_* a, \tag{2.28}$$

where $y \in Z^1(K, R^4)$, $a \in C^1(K, U)$ and $d \in R^4$. Of course, the systems $\bar{u}$ and u are equivalent if and only if $y \in B^1(K, R)$. The difference $\pi_*(\bar{u} - u)$, however, necessarily belongs to the kernel of ∂_*. It is instructive to show this quite explicitly.

Let us therefore characterize Ker ∂_*. Take an element $x \in Z^1(K, R^4/U)$ such that $[x] \in \operatorname{Ker} \partial_*$. There is then an $y \in Z^1(K, R^4)$ such that

$$[x] = [\pi_* y].$$

Therefore, we may choose a $d \in R^4$ such that

$$x = \pi_* y + \delta \pi d$$

or

$$x = \pi_*(y + \delta d). \tag{2.29}$$

This relation then characterizes the elements $x \in Z^1(K, R^4)$ whose cohomology class $[x]$ lies in Ker ∂_*. It is now easy to see that $\pi_*(\bar{u} - u)$ has the above form.

Let us finally remark that all the above considerations are useful only if G is an inhomogeneous subgroup of $IO(3, 1)$, *i.e.* if it contains transformations without fixpoint. Note that this means not $U \neq 0$, but $[\pi_* u] \neq 0$. If there are no primitive translations, a system of non-primitive translations is a 1-cocycle: $u \in Z^1(K, R^4)$.

To conclude, we want to establish an important property of the subgroups of the Poincaré group .

Proposition 1. Let G be a subgroup of $IO(3,1)$ such that U, defined by $U = R^4 \cap G$, generates R^4 as real vector space. Let V be a normal abelian subgroup of G. Then

$$V \subset U.$$

Proof. Write the elements of G as $\langle a, \alpha \rangle$ with $a \in U$, $\alpha \in K \simeq G/U$. Take any $v = \langle a, \alpha \rangle \in V \subset G$.

Then

$$w = \langle b, \varepsilon\rangle\langle a, \alpha\rangle\langle b, \varepsilon\rangle^{-1} = \langle a + b - \alpha b, \alpha\rangle \in V$$

for any $b \in U$, because V is normal in G. Furthermore

$$v^{-1}wv = \langle -m(\alpha^{-1}, \alpha) - \alpha^{-1} a, \alpha^{-1}\rangle\langle a + b - \alpha b, \alpha\rangle\langle a, \alpha\rangle$$
$$= \langle a - b + \alpha^{-1} b, \alpha\rangle = w,$$

because V is abelian. Therefore

$$2b = \alpha b + \alpha^{-1} b, \qquad \forall b \in U. \tag{2.30}$$

Consider now the scalar product $b^{\mathrm{T}}gb$ of b with b, and take into account that $\alpha^{-1} = g\alpha^{\mathrm{T}}g$.

$$2b^{\mathrm{T}}gb = b^{\mathrm{T}}g(\alpha b) + b^{\mathrm{T}}g(g\alpha^{\mathrm{T}}gb) = b^{\mathrm{T}}g(\alpha b) + (\alpha b)^{\mathrm{T}}gb = 2b^{\mathrm{T}}g(\alpha b). \tag{2.31}$$

From this we derive that, for arbitrary b, the vectors b and αb are orthogonal to $\alpha b - b$, and that the latter is a lightlike vector. Let us choose b as timelike vector. Then this timelike vector is orthogonal to the lightlike vector $\alpha b - b$. This is impossible unless $\alpha b - b = 0$.

Thus necessarily for any timelike $b \in U$

$$\alpha b = b.$$

According to the hypotheses U contains four linearly independent timelike elements. Therefore, $\alpha = \varepsilon$, *i.e.* $V \subset U$.

Corollary. A normal, free abelian and maximal abelian subgroup $U \subset G$ is unique.

Proof. Let U and V be two such subgroups. Then from the preceding proposition $U \subset V$ and also $V \subset U$, so that $U = V$.

Note that in spaces with index larger than one, timelike vectors may be orthogonal to lightlike vectors. (The index of a space with a metric is the dimension of its maximal lightlike – *i.e.* isotropic – subspaces). It can be shown that in such spaces (2.30) does not imply $\alpha = \varepsilon$.

3. *Imbeddings.* Up to now we have considered subgroups G of $IO(3,1)$. Now we envisage the situation where only a subgroup $U \subset R^4$ and a subgroup $K \subset \mathrm{Aut}\, U \cap O(3,1)$ are given. In view of the preceding discussions we construct groups in two ways from U and K, and investigate the conditions under which such groups can be imbedded as subgroups into $IO(3,1)$.

A first way is to form the extensions of U by K. This corresponds to choosing a 2-cocycle $m \in Z^2(K, U)$, which we shall normalize by putting $m(\alpha, \varepsilon) = m(\varepsilon, \alpha) = 0$ for any $\alpha \in K$. The elements of the group are the pairs

$$\langle a, \alpha\rangle, \qquad a \in U, \qquad \alpha \in K \tag{3.1}$$

and the multiplication is defined as

$$\langle a, \alpha\rangle\langle b, \beta\rangle = \langle a + \alpha b + m(\alpha, \beta), \alpha\beta\rangle. \tag{3.2}$$

Groups given in this manner are noted

$$G_1 = \{U, K, m\}. \tag{3.3}$$

We know from refs. 11 or 12 that $G_1 \subset IO(3,1)$ if and only if

$$[\iota_*][m] = 0, \tag{3.4}$$

where $[m]$ is the cohomology class of m, and $[\iota_*]$: $H^2(K, U) \to H^2(K, R^4)$ is induced by the imbedding ι: $U \to R^4$.

A second way of constructing groups from U and K is the following: We choose a normalized 1-cochain $u \in C^1(K, R^4)$, *i.e.* one with the property $u(\varepsilon) = 0$. Furthermore we require

$$\pi_* u \in Z^1(K, R^4/U). \tag{3.5}$$

Then relation (2.23) holds. We now consider the set G_2 defined by

$$G_2 = \{[a + u(\alpha), \alpha] \mid a \in U, \alpha \in K\} \tag{3.6}$$

and introduce a multiplication by

$$[a + u(\alpha), \alpha][b + u(\beta), \beta] = [a + \alpha b + u(\alpha) + \alpha u(\beta), \alpha\beta]. \tag{3.7}$$

Note that, owing to (2.23), the right-hand side has the correct form $[c + u(\alpha\beta), \alpha\beta]$. It is now easy to verify that the set (3.5) with multiplication (3.6) forms a group. In particular

$$[a + u(\alpha), \alpha]^{-1} = [-\alpha^{-1} a - \alpha^{-1} u(\alpha), \alpha^{-1}]$$

and again the right-hand side has the correct form because (2.23) implies

$$-\alpha^{-1} u(\alpha) \equiv u(\alpha^{-1}) \qquad (\text{mod } U).$$

Groups constructed in this way are noted

$$G_2 = \{U, K, u\}. \tag{3.8}$$

Let us choose a monomorphic section $\bar{r}$: $O(3,1) \to IO(3,1)$ and write the elements of $IO(3,1)$ as in (2.3). The mapping μ: $G_2 \to IO(3,1)$ defined by

$$\mu[a + u(\alpha), \alpha] = (a + u(\alpha), \alpha) \tag{3.9}$$

is then obviously a monomorphism. Note that in fact the group G_2 is already, by construction, a subgroup of $IO(3,1)$. Indeed condition (3.4) is equivalent to the existence of $u \in C^1(K, R^4)$ such that (3.5) holds and, *vice versa*, if (3.5) holds, then $[m]$ defined by

$$[m] = \partial_*[\pi_* u] \tag{3.10}$$

has manifestly the property (3.4).

Since

$$U \lhd G_2, \qquad G_2/U \simeq K \tag{3.11}$$

the group G_2 is an extension of U by K. The cohomology class of the extension is precisely that given by (3.10). Groups $G_2' = \{U, K, u'\}$ and $G_2'' = \{U, K, u''\}$ give rise to equivalent extensions, *i.e.* to the same $[m] \in H^2(K, U)$ if and only if they are related as in (2.28):

$$u'' - u' \in Z^1(K, R^4). \tag{3.12}$$

The properties of a subgroup $G \subset IO(3, 1)$ as group of transformations are given by its system, u, of non-primitive translations. Equivalent systems of non-primitive translations describe, up to a choice of origin, the same group of transformations. It is therefore natural for us to identify such groups. Inequivalent systems of non-primitive translations give rise to different groups of transformations. After the above identification, there is a one-to-one correspondence between the elements of $H^1(K, R^4/U)$ and groups of transformations G with a group of primitive translations U and a point group K. The order of Ker ∂_* gives the number of different groups of such transformations that give rise to equivalent extensions of U by K.

If $H^1(K, R^4) = 0$, as in euclidean crystallography, then ∂_* is a monomorphism, and inequivalent systems of non-primitive translations give rise to inequivalent extensions.

We now consider a slightly more general situation.

Proposition 2. Let there be given a group G and a homomorphism λ: $A \to R^4$, a group B and homomorphisms ν: $B \to O(3,1)$, φ: $B \to$ Aut A. Consider an extension G of A by B with cohomology class $[m]$. Choose sections $r: B \rightarrow\!\!- G$ and $\bar{r}: O(3,1) \rightarrow\!\!- IO(3,1)$. The group G can be imbedded by a monomorphism μ: $G \to IO(3,1)$ defined as

$$\mu\langle a, \alpha\rangle = (\lambda a + u(\alpha), \nu\alpha) \tag{3.13}$$

with $u(\varepsilon) = 0$ if and only if

(i) $\lambda[(\varphi\alpha)\, a] = (\nu\alpha)(\lambda a)$; (3.14)

(ii) $[\lambda_*][m] = 0$; (3.15)

(iii) λ is a monomorphism (3.16)

(iv) Ker $\nu \cap u^{-1}(\lambda A) = \varepsilon$. (3.17)

Proof. We know from refs. 11 and 12 that (i) and (ii) are necessary and sufficient conditions for the existence of a homomorphism μ. Consider now $\langle a, \alpha\rangle \in$ Ker μ. Then $\alpha \in$ Ker $\nu \cap u^{-1}(\lambda A)$. If μ is a monomorphism, then (iii) follows because λ is the restriction of μ to A. Furthermore (iv) follows.

Conversely if (iii) and (iv) hold, then $\alpha = \varepsilon$, $u(\varepsilon) = 0$ and $a = 0$. Thus μ is a monomorphism.

Thus there may exist subgroups G of the Poincaré group, having "point groups" $B \simeq G/A$ that are not isomorphic to subgroups of the point group of the Poincaré group (*i.e.* the Lorentz group). However, in these cases B does not operate faithfully on A.

Corollary 1. Under the conditions of proposition 2

$$\operatorname{Ker} \nu \subset \operatorname{Ker} \varphi.$$

Proof. The proof is based on condition (3.14). Suppose $\alpha \in \operatorname{Ker} \nu$. Then, since λ is a monomorphism, $(\varphi\alpha)\, a = a$ and $\alpha \in \operatorname{Ker} \varphi$.

The question under what circumstances φ and ν are monomorphisms is elucidated by the following.

Corollary 2. If λA generates R^4 as real vector space, then A is maximal abelian in G.

Proof. See proposition 2, of ref. 11.

Since by proposition 1 of the same reference we know that A is maximal abelian in G if and only if φ is a monomorphism, the question is now answered.

4. *A generalized Bieberbach theorem.* Bieberbach[5]) has shown in the case of n-dimensional euclidean crystallography that two isomorphic abstract space groups can always be imbedded as conjugate subgroups into the affine group $A(n)$. We shall now show that the validity of a generalized Bieberbach theorem requires the vanishing of the first cohomology group $H^1(K, R^n)$.

We consider the Poincaré group as subgroup of $A(4)$.

Proposition 3. Let G and G' be two subgroups of the Poincaré group: $G \subset IO(3,1) \subset A(4)$, $G' \subset IO(3,1) \subset A(4)$ with the following properties:

(i) The normal subgroups $U \lhd G$, $U' \lhd G'$ defined by

$$U = R^4 \cap G, \qquad U' = R^4 \cap G' \tag{4.1}$$

generate R^4 as real vector space.

(ii) The point groups K and K' defined by

$$G/U \simeq K \subset O(3,1) \subset GL(4, R),$$
$$G'/U' \simeq K' \subset O(3,1) \subset GL(4, R), \tag{4.2}$$

have vanishing first cohomology groups with coefficients in R^4

$$H^1(K, R^4) = H^1(K', R^4) = 0. \tag{4.3}$$

Then G and G' are isomorphic if and only if they are conjugate in $A(4)$.

Proof. It is clear that conjugate subgroups are isomorphic. We have only to show that the isomorphic subgroups G and G' are also conjugate.

Denote by ψ the isomorphism $\psi: G \to G'$. Let λ be the restriction of ψ to U. Then λ is a monomorphism. But λU is a normal abelian subgroup of G', it is free abelian and generates R^4. Therefore, by proposition 1, $\lambda U = U'$, *i.e.* λ is an isomorphism. Then the induced homomorphism $\omega: K \to K'$ is also and isomorphism. We thus have a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \to & U & \to & G & \to & K & \to & 1 \\ & & \lambda\downarrow & & \downarrow\psi & & \downarrow\omega & & \\ 0 & \to & U' & \to & G' & \to & K' & \to & 1 \end{array} \tag{4.4}$$

with isomorphisms λ, ψ, ω.

Since U and U' generate R^4, λ may be extended by linearity to an automorphism $\chi \in GL(4, R)$ of the real vector space R^4:

$$\begin{array}{ccc} U & \xrightarrow{\iota} & R^4 \\ \lambda\downarrow & & \downarrow\chi \\ U' & \xrightarrow[\iota]{} & R^4 \end{array} \tag{4.5}$$

From (4.4) we deduce

$$\lambda(\alpha a) = (\omega\alpha)(\lambda a), \qquad \forall a \in U \tag{4.6}$$

or, by extension to R^4,

$$\chi(\alpha d) = \omega\alpha(\chi d), \qquad \forall d \in R^4,$$

which can be written

$$\omega\alpha = \chi\alpha\chi^{-1}, \qquad \forall \alpha \in K \subset GL(4, R). \tag{4.7}$$

Thus K and K' are conjugate in $GL(4, R)$.

Another consequence of (4.4) is

$$[\lambda_*][m] = [\omega^*][m'] \in H^2(K, U'), \tag{4.8}$$

where $[m]$ is the cohomology class of G and $[m']$ that of G'. As in (2.16) we may, however, always choose 2-cocycles $m' \in [m']$ such that even

$$\lambda_* m = \omega^* m' \in Z^2(K, U') \tag{4.9}$$

holds. The fact that G and G' are subgroups of $IO(3,1)$ gives rise to the relations

$$[\iota_*][m] = 0 \in H^2(K, R^4), \qquad [\iota_*][m'] = 0 \in H^2(K', R^4) \tag{4.10}$$

or

$$\iota_* m = \delta u, \qquad \iota_* m' = \delta' u'. \tag{4.11}$$

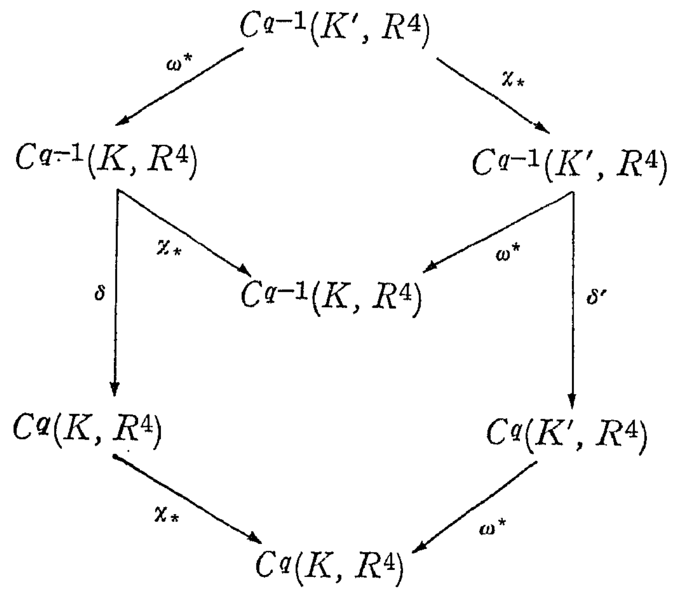

Fig. 1

Now

$$\chi_*\delta u = \chi_*\iota_* m = \iota_*\lambda_* m = \iota_*\omega^* m' = \omega^*\iota_* m' = \omega^*\delta' u'. \tag{4.12}$$

From (4.7) we deduce the commutation relations

$$\omega^*\delta'\chi_* = \chi_*\delta\omega^* \quad \text{or} \quad \omega^*\delta'\omega_*{}^{-1} = \chi_*\delta\chi_*{}^{-1}, \tag{4.13}$$

valid on q-cochains with $q \geq 1$ and corresponding to the commutative diagram shown in fig. 1.
For $q = 0$ we have the simpler relation

$$\omega^*\delta'\chi = \chi_*\delta. \tag{4.14}$$

Using (4.13) we now transform (4.12):

$$\omega^*\delta'\omega^{*-1}\,\chi_* u = \omega^*\delta' u'.$$

Thus

$$\omega^{*-1}\,\chi_* u - u' \in Z^1(K', R^4) = B^1(K', R^4)$$

and there exists an element $d \in R^4$ such that

$$\chi_* u + \omega^*\delta' d = \omega^* u'. \tag{4.15}$$

For a better comparison with (4.12), we may put $d = \chi f$ and use (4.12) to transform (4.15) into

$$\chi_*(u + \delta f) = \omega^* u'. \tag{4.16}$$

Let $\mu g = \mu\langle a, \alpha\rangle = (a + u(\alpha), \alpha)$ define the imbedding of G into $A(4)$

and

$$\mu'\psi g = \mu' g' = \mu'\langle \lambda a, \omega\alpha\rangle' \\ = (\lambda a + u'(\omega\alpha), \omega\alpha) = (\lambda a + (\omega^* u')(\alpha), \omega\alpha)$$

that of $G' = \psi G$ into $A(4)$. Then by (4.7) and (4.15)

$$(d, \chi)\, \mu g (d, \chi)^{-1} = (\chi a + \chi u(\alpha) + (\delta' d)(\chi\alpha\chi^{-1}), \chi\alpha\chi^{-1}) \\ = (\lambda a + (\omega^* u')(\alpha), \omega\alpha) = \mu' g'. \tag{4.17}$$

Note that in this proof we have used only $H^1(K', R^4) = 0$, but $H^1(K', R^4)$ is isomorphic to $H^1(K, R^4)$ by $[\omega^*]$ so that it is sufficient to suppose that one of the two cohomology groups (4.3) vanishes. Note also that the property that K' be a subgroup of $O(3,1)$ has been used in the proof through the use of proposition 1. But then K, which is conjugate in $A(4)$ to K', is also a subgroup of $O(3,1)$.

We now want to find a natural equivalence relation allowing the identification of a number of groups; this is the prerequisite of any reasonable classification. In the case of space groups, Bieberbach identified isomorphic groups, whereas, according to Frobenius [6]), it was better, from the crystallographic point of view, to identify only groups that are conjugate subgroups of the affine group. Bieberbach [5]) has shown, for the case of a definite metric, that the two corresponding classifications coincide. We have, however, just seen that this is not necessarily so in the relativistic case: affine conjugation provides a finer classification than does plain isomorphism. From the physical point of view the classification by affine conjugation is still not fine enough, since the spacelike, timelike or lightlike character of translations is not conserved. A classification according to conjugation classes of the Poincaré group is, however, already too fine. Indeed it is reasonable not to distinguish groups that are related by dilatations of R^4. Dilatations are transformations that preserve parallelism. They form a group $D(4)$ that contains (i) the translations $T \simeq R^4$, and (ii) the central dilatations C, also called homotethies or scalings. The group C is the center of $GL\ (4, R)$, *i.e.* consists of scalar matrices. The dilatations form a normal subgroup of the affine group: $D(4) \lhd A(4)$. Furthermore $R^4 \lhd D(4)$ and $D(4)/R^4 = C$.

These considerations prompt us to adopt the following definitions of equivalence.

Definition. Two subgroups G and G' of $IO(3,1)$ – with translation subgroups $U = R^4 \cap G$ and $U' = R^4 \cap G'$ respectively, which generate R^4 as real vector space – are equivalent if there is an isomorphism $\psi: G \to G'$ whose restriction χ to U has the following form

$$\chi = \zeta\chi_0, \qquad \zeta \in C, \qquad \chi_0 \in O(3,1). \tag{4.18}$$

The isomorphism χ not only maps translations onto translations but also conserves the character (timelike, ...) of the translations, and the latter property could have been used, instead of (4.18) in the definition of equivalence. Note also that for equivalent G and G' the corresponding point groups K and K' are related by

$$K' = \chi K \chi^{-1} = \chi_0 K \chi_0^{-1}. \tag{4.19}$$

In the more general case, when neither U nor U' does generate the real vector space R^4, some precautions must be taken in the definition of equivalence. Firstly, it is not sufficient to ask that translations of a given character be mapped (isomorphically) on translations of the same character. Indeed U may be, for instance, a two-dimensional lattice generated by two spacelike translations e_1 and e_2

$$U = \{n_1 e_1 + n_2 e_2 \,|\, n_1, n_2 \in Z\}$$

and U' the set of vectors generated by the two parallel translations e_1 and ρe_1 where ρ is any irrational number:

$$U' = \{n_1 e_1 + n_2 \rho e_1 \,|\, n_1, n_2 \in Z\} \simeq Z^2.$$

Then U and U' are isomorphic and contain only spacelike vectors and still they represent two (geometrically and physically) different situations that we would not like to identify.

Definition. Two subgroups G and G' of $IO(3,1)$ with nonzero translations groups $U = R^4 \cap G$ and $U' = R^4 \cap G'$ respectively, are equivalent if there is an isomorphism $\psi\colon G \to G'$, whose restriction λ to U has the form $\lambda = \xi\lambda_0$ with $\xi \in C$ and λ_0 the restriction of some $\chi_0 \in O(3,1)$ from R^4 to U.

Secondly, if $U = 0$ the restriction λ to U does not induce a conjugation of K in $O(3.1)$. This is, however, what we must require for equivalent groups. Otherwise we would be led to identify for instance the following two groups: (i) K generated by a rotation of one radian around a given (spacelike) axis followed by a translation along this axis, and (ii) K' generated by a pure Lorentz transformation (boost) in a given space direction followed by a translation in that direction. Again this would not be desirable.

Definition. Two inhomogeneous subgroups G and G' of $IO(3,1)$ containing no primitive translations ($U = U' = 0$) are equivalent if their point groups (to which they are, by the way, isomorphic) are conjugate subgroups of $O(3,1)$.

5. *Examples.* By way of illustration, a number of physical systems having as symmetry group a inhomogeneous subgroup of the Poincaré group are indicated and briefly discussed.

(i). The static crystal. In space–time, the group of primitive translations of a static crystal consists of discrete lattice translations combined with continuous time translations. The point group is a crystallographic Shubnikov group.

(ii). The transverse electromagnetic plane wave[16]) (in empty space). Consider a TEM-wave propagating in the e_3 direction with wave vector $k = (2\pi/\lambda)(e_0 + e_3)$ (we put $c = 1$). The group of primitive translations is generated by infinitesimal translations in the e_1, e_2 and $e_0 + e_3$ directions and by a discrete translation λe_3. The point group K and a system of non-primitive translations u depend on the polarization of the TEM-wave.

For linearly polarized waves, there are in K mirrors and continuous Lorentz transformations. Furthermore $u(K)$ contains an infinite number of non-primitive translations inequivalent to zero, which can all be chosen to amount to half a wavelength in the e_3 direction.

For circulary polarized TEM-wave, continuous rotations $R_3(\theta)$ by an angle θ around the e_3 axis are associated with continous non-primitive translations along the same axis:

$$u(R_3(\theta)) = \pm\,(\theta\lambda/2\pi)\,e_3, \qquad 0 \leq \theta < 2\pi, \tag{5.1}$$

the plus sign applying, if the wave is right-hand and the minus sign if it is left-hand circularly polarized.

(iii). The diffracting crystal[17]),[18]). The system is here a static crystal diffracting an incident electromagnetic monochromatic wave (in the geometrical approximation). The symmetry of such a system is a minkowskian crystallographic group; in fact the group of primitive translations U is discrete and generates a 4-dimensional lattice in space and time. The point group is a Shubnikov group and depends on the polarization of the incident radiation. The question whether the whole symmetry group is simply the intersection of the symmetry group of the static crystal with that of the incident radiation is under investigation.

(iv). Uniform electromagnetic fields[19]) (in empty space). In this case all translations in space and time are primitive ($U = R^4$); the non-primitive translations are equivalent to zero and the symmetry group can be presented as semi-direct product of R^4 by a point group K. This latter depends on the relative orientation and magnitude of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In all the cases K, contains discrete as well as continuous Lorentz transformations. The algebras of these groups have been investigated independently by another team[20]) with a view to classifying electromagnetic form factors of elementary particles.

(v). Normal modes in wave guides and resonant cavities[21]) (in empty space). A number of symmetry groups of TE-and TM-modes in wave guides and cavities has been determined. One of the most interesting features is observed in the symmetry groups of propagating modes, where mirrors occur in a frame of reference moving with the same relative velocity (with respect to the laboratory system) as the propagation velocity. Some of these elements give rise to relativistic symmetries (so-called Lorentz mirrors) in the laboratory system.

At the moment we are looking for other systems having symmetries in space-time. The physical consequences of these symmetry groups are also under investigation.

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