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KINETO-ELECTRIC AND KINETOMAGNETIC EFFECTS IN CRYSTALS†

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The kineto-electric, kinetomagnetic and ferrokinetic effects are defined as corresponding to the following terms in the density of stored free enthalpy g of the crystal

$$g = \ldots + \eta_{ik} v_i E_k + \zeta_{ik} v_i cB_k + {}^0p \cdot v$$

and the symmetry conditions for the existence of these effects are discussed. A corresponding relativistic Lagrangian density $\mathscr{L}$ is set up

$$\mathscr{L} = \frac{1}{8c} \chi^{\alpha\beta\sigma\delta} F_{\alpha\beta} F_{\sigma\delta} + \tfrac{1}{2}\xi^{\sigma\alpha\beta} u_\sigma F_{\alpha\beta} + \tfrac{1}{2}\hat{\rho}^{\alpha\beta} u_\alpha u_\beta$$

and from this the electric polarization $\mathbf{P}$, the magnetization $\mathbf{J}$ and the linear momentum $\mathbf{p}$ of a crystal moving in an electromagnetic field are determined as

$$c\mathbf{P}_\perp = \frac{1}{\gamma}(cP + \beta \times J)_\perp, \quad \mathbf{J}_\perp = \frac{1}{\gamma}(J - \beta \times cP)_\perp, \quad c\mathbf{P}_\parallel = P_\parallel, \quad \mathbf{J}_\parallel = J$$

$$\mathbf{p}_\parallel = -\left[\frac{1}{c^2} E \times J + P \times B + ({}^2p^0 - {}^3p^0)\beta\right]_\parallel$$

$$\mathbf{p}^0 = \frac{1}{|\beta|}\left[\frac{1}{c^2} E \times J + P \times B + {}^2p + {}^3p\right]_\parallel$$

The expressions for P, J, ${}^2p$, ${}^3p$ in terms of the applied fields E, B, and v are given in the text.

The expression kineto-electric effect denotes the occurrence of electric polarization P in a moving crystal, proportional to the velocity v and in absence of any applied electromagnetic field: $P \sim v$. Similarly, kinetomagnetic corresponds to $J \sim v$. Besides these we shall consider also bilinear effects such as $P \sim vE$, $J \sim vcB$.

For several reasons, but especially with a view to the relativistic treatment, which we shall undertake, it appears convenient to consider cB as the magnetic field and to measure the magnetization by J. Let us recall the relevant relations:

$$D = \varepsilon_0 E + P$$

$$H = \frac{1}{\mu_0} B - J \quad M = \mu_0 J.$$

To understand the kinetic effects we want to discuss, it is useful to distinguish four types of vector, and not only two; viz.:

type M "axial"
type P "polar"
type j "axio-polar"
type $\dot{M}$

These four types correspond to the four irreducible representations of the dihedral group $\bar{1}1'$ of order four generated by the space inversion $\bar{1}$ and the time reversal $1'$:

$\bar{1}1'$	1	$\bar{1}$	$1'$	$\bar{1}'$	
	1	1	1	1	$\dot{M}$, grad P
	1	1	−1	−1	M, grad v
	1	−1	1	−1	P, $\dot{v}$
	1	−1	−1	1	j, v, A, $\dot{P}$, grad M

We shall here adopt the physical interpretation of velocity for the "axio-polar" vector.

Ferro-electricity, ferromagnetism and magneto-electricity correspond to the existence, in the density of stored free enthalpy g for a single domain, of terms proportional to the electric field E to the magnetic field B and to the product of both

$$-g(E, B) = \ldots + {}^0P \cdot E + {}^0J \cdot B + \varepsilon_0 \lambda_{ik} E_i cB_k$$

By stored free enthalpy we mean the free enthalpy where the contributions of the vacuum have been

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subtracted. Here ${}^0P$ and ${}^0J$ denote the spontaneous polarization and magnetization, and $\varepsilon_0 \lambda_{ik}$ the magneto-electric coefficient in the variables and units we have chosen. (λ will be defined below.) These properties are described by polar and axial vectors. If we want to consider the analogous properties corresponding to the axio-polar vector v, we have to introduce the following terms into the density of free enthalpy

$$-g(E, B, v) = \ldots + {}^0p \cdot v + \eta_{ik} v_i E_k + \zeta_{ik} v_i cB_k$$

and we shall denominate the effects described by these terms as ferrokinetic, kineto-electric and kinetomagnetic; we find indeed:

$$P_i = -\frac{\partial g}{\partial E_i} = \ldots + \eta_{ki} v_k$$

$$J_i = -\frac{\partial g}{\partial B_i} = \ldots + c\zeta_{ki} v_k$$

$$p_i = -\frac{\partial g}{\partial v_i} = \ldots + {}^0p_i + \eta_{ik} E_k + \zeta_{ik} cB_k$$

Here p_i of course denotes (linear) momentum. The ferrokinetic effect corresponds to the existence of momentum without applied fields E, B, v.

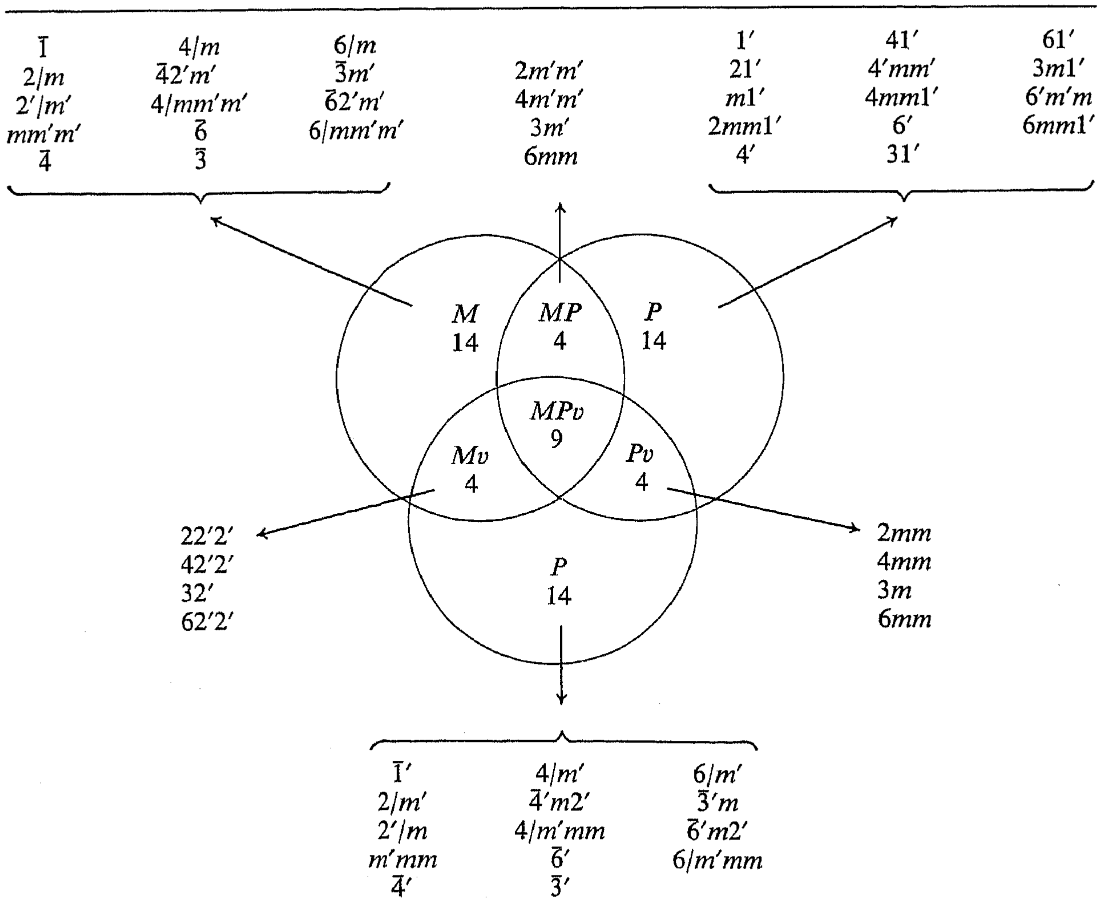
Formally these new effects are completely analogous to the old ones. There are 31 (out of the 122) Shubnikov groups that are compatible with the ferrokinetic effect as is the case for ferro-electricity and ferromagnetism. These groups are:[1]

> 1, 2, *m*, 2′, *m*′, 1̄′, *mm*2, 22′2′, 2/*m*′, 2′/*m*, *mm*′/2′, *mmm*′, 4, 4̄′, 4/*m*′, 4*mm*, 42′2′, 4̄′2′*m*, 4/*m*′*mm*, 3, 6, 6̄′, 3̄′, 6/*m*′, 3*m*, 32′, 3̄′*m*, 6*mm*, 62′2′, 6̄′*m*2′ and 6/*m*′*mm*.

(see also Table I).

Similarly, there are 58 Shubnikov groups compatible with the kineto-electric and kinetomagnetic effect respectively as is the case for the magneto-electric effect. In the three listings of 58 groups, in Table II, 36 are in common, i.e. they are compatible with the three effects. In each listing, the remaining 22 groups permit no other of the three considered effects; in Table II they are underlined. It can be seen

TABLE I



MPv: 1, 2, *m*, 2′, *m*′, *mm*′2′, 4, 3, 6 that for each effect there are altogether 11 types of tensorial representations. The type of the kineto-electric tensor η_{ik} or the kinetomagnetic tensor ζ_{ik} is that of the magneto-electric tensor λ_{ik} for the corresponding group which can be easily found (see e.g. Ref. 2). In Table II these groups are listed on corresponding places.

To conclude this brief excursion into the magic land of symmetry we show in Figure 1 the number of groups that permit a given combination of the 6 effects: ferromagnetism (M), ferro-electricity (P), ferrokinetic effect (p), magneto-electric effect (α), kineto-electric effect (η) and kinetomagnetic effect (ζ). In Table III, the corresponding groups are explicitly listed. Note that, out of the 64 imaginable combinations of 6 effects, only 15 are possible.

When speaking about kineto-electric and kineto-magnetic effects it is always necessary to take into account special relativity, because the Lorentz transformation gives rise to kinetic effects. More precisely it gives rise to second-order kinetic effects, namely to polarizations and magnetizations bilinear in the velocity and the electromagnetic field. This is of course well known, but we shall see it once more later on in our formulae. Furthermore, we know that there are no symmetry conditions for the existence of this effect. However, when certain conditions are fulfilled, viz. if the crystal is ferro-electric or ferromagnetic, or else ferromagento-electric, we observe a polarization, or magnetization, that is proportional to the velocity, and this is precisely a kineto-electric or kineto-magnetic effect. We shall therefore proceed relativistically.

We consider a density of free enthalpy—or a lagrangian density $\mathscr{L}(F.., u.)$, since we are not interested in thermal properties—composed of three parts

$$\mathscr{L}(F.., u.) = {}^1\mathscr{L}(F.., u.) + {}^2\mathscr{L}(F.., u.) + {}^3\mathscr{L}(u.)$$

The first part

$${}^1\mathscr{L}(F.., u.) = \frac{1}{8c}\chi^{\alpha\beta\sigma\delta} F_{\alpha\beta} F_{\sigma\delta}$$

is symmetry independent and well known. We shall however recall the meaning of the symbols and the notations and conventions we use.

The metric tensor is

$$g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1) = g^{\alpha\beta}$$

so that the coordinates of an event are

$$x^\alpha = (ct, x_1, x_2, x_3)$$

The four-velocity is

$$u^\alpha = \frac{1}{\gamma}(c, v_1, v_2, v_3) = \frac{c}{\gamma}(1, \beta_1, \beta_2, \beta_3)$$

TABLE II

	$E_l B_k$ λ	$v_l E_k$ ξ	$v_l B_k$ ξ
1	$1, \underline{\bar{1}'}$	$1, \underline{\bar{1}}$	$1, \underline{1'}$
2	$2, m', \underline{2'/m}$	$2, m', \underline{2/m}$	$2, m', \underline{21'}$
3	$m, 2', \underline{2'/m}$	$m, 2', \underline{2'/m'}$	$m, 2', \underline{m1'}$
4	$222, 2'm'm', \underline{m'm'm'}$	$222, 2m'm', \underline{mmm}$	$222, 2m'm', \underline{2221'}$
5	$2mm, 22'2', 2'mm', \underline{m'mm}$	$2mm, 22'2', 2'mm', \underline{mm'm'}$	$2mm, 22'2', 2'mm', \underline{mm21'}$
6	$4, \bar{4}', \underline{4/m'}$ $3, 6, \underline{\bar{6}'}, \underline{\bar{3}'}, \underline{6/m'}$	$4, \bar{4}', \underline{4/m}$ $3, 6, \underline{\bar{6}}, \underline{\bar{3}}, \underline{6/m}$	$4, \bar{4}', \underline{41'}$ $3, 6, \underline{6'}, \underline{31'}, \underline{61'}$
7	$\bar{4}, 4', \underline{4'/m'}$	$\bar{4}, 4', \underline{4'/m}$	$\bar{4}, 4', \underline{\bar{4}1'}$
8	$422, \bar{4}'2m', 4m'm'$ $\underline{4/m'm'm'}, 32, 3m', 622$ $\underline{\bar{3}'m'}, \underline{\bar{6}'m'2}, 6m'm', \underline{6/m'm'm'}$	$422, \bar{4}'2m', 4m'm'$ $\underline{4/mmm}, 32, 3m', 622$ $\underline{\bar{3}m}, \underline{\bar{6}m2}, 6m'm', \underline{6/mmm}$	$422, \bar{4}'2m', 4m'm'$ $\underline{4221'}, 32, 3m', 622$ $\underline{321'}, \underline{6'2'2}, 6m'm', \underline{62221'}$
9	$\bar{4}2m, 4'22', \bar{4}2'm', 4'm'm$ $\underline{4'/m'm'm}$	$\bar{4}2m, 4'22', \bar{4}2'm', 4'm'm$ $\underline{4'/mmm'}$	$\bar{4}2m, 4'22', \bar{4}2'm', 4'm'm$ $\underline{\bar{4}2m1'}$
10	$4mm, 42'2', \bar{4}'m2'$ $\underline{4/m'mm}, 3m, 32', \underline{\bar{3}'m}$ $6mm, 62'2', \underline{\bar{6}'m2'}, \underline{6/m'mm}$	$4mm, 42'2', \bar{4}\,m2'$ $\underline{4/mm'm}, 3m, 32', \underline{\bar{3}m'}$ $6mm, 62'2', \underline{\bar{6}m'2'}, \underline{6/mm'm'}$	$4mm, 42'2', \bar{4}'m2'$ $\underline{4mm1'}, 3m, 32', \underline{3m1'}$ $6mm, 62'2', \underline{6'mm'}, \underline{6mm1'}$
11	$23, \underline{m'3}, 432, \underline{\bar{4}'3m'}, \underline{m'3m'}$	$23, \underline{m3}, 432, \underline{\bar{4}3m}, \underline{m3m}$	$23, \underline{231'}, 432, \underline{4'32'}, \underline{4321'}$

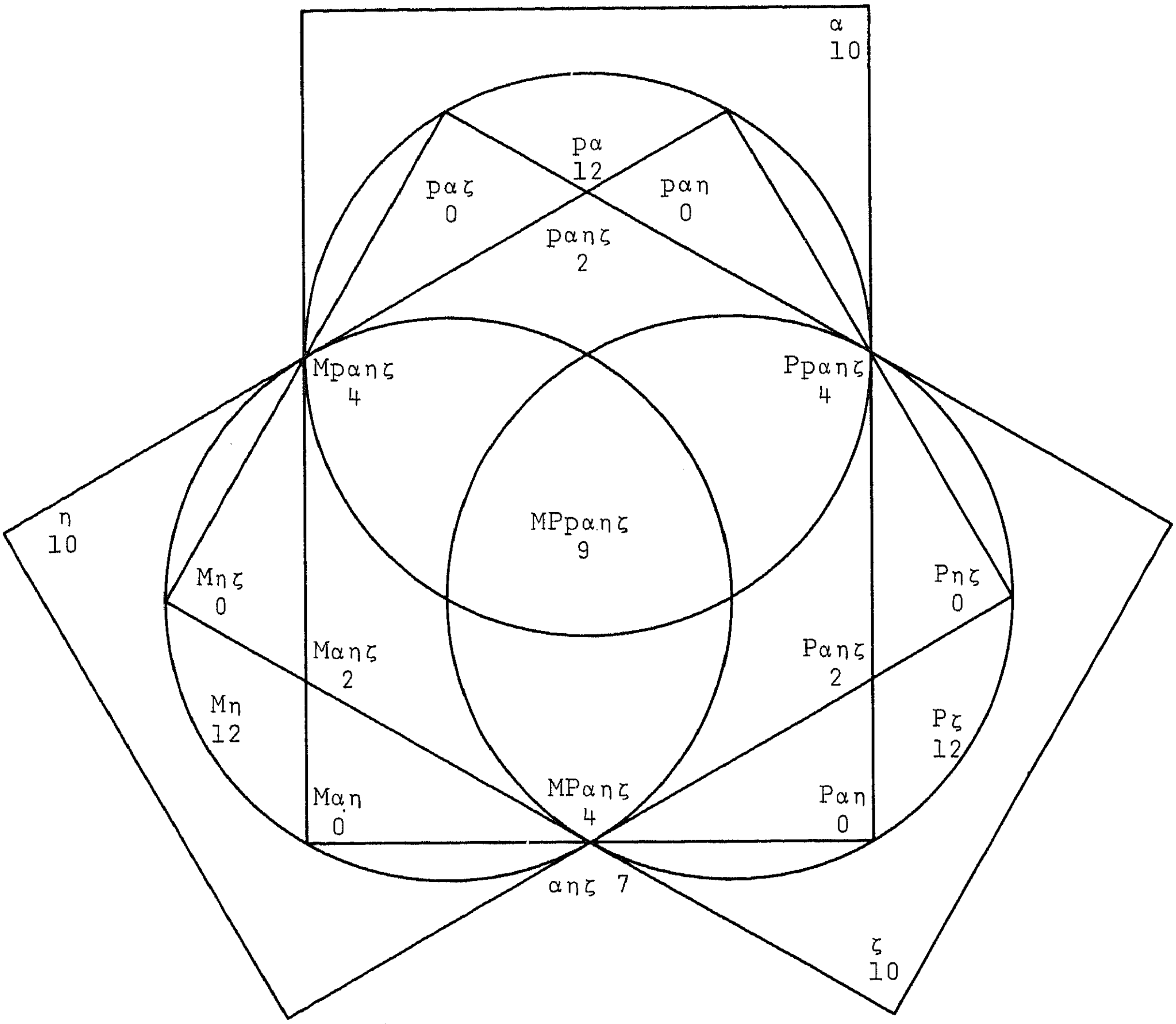


FIGURE 1

with

$$\gamma^2 = 1 - \beta^2, \quad \beta = \frac{v}{c}\,.$$

The field tensor is

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{pmatrix}$$

its physical dimension is V/m. The polarization tensor is

$$P^{\alpha\beta} = \begin{pmatrix} 0 & cP_1 & cP_2 & cP_3 \\ -cP_1 & 0 & -J_3 & J_2 \\ -cP_2 & J_3 & 0 & -J_1 \\ -cP_3 & -J_2 & J_1 & 0 \end{pmatrix}$$

with dimension A/m. These tensors are related by the constitutive equation

$$P^{\alpha\beta} = \tfrac{1}{2}\chi^{\alpha\beta\sigma\delta} F_{\sigma\delta}$$

equivalent to the following tensor equations

$$cP = \frac{1}{R_0}\,[\kappa^B E + \lambda cB]$$

$$J = \frac{1}{R_0}\,[\tilde{\lambda} E - \varphi^E cB]$$

Here

$$\frac{1}{R_0} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2} = 2.654 \times 10^{-3}\,\mathrm{A/V}$$

and the tilde denotes the transpose.

TABLE III

$MP\rho\alpha\eta\zeta$	9	$1, 2, m, 2', m', mm'2', 4, 3, 6$
$P\rho\alpha\eta\zeta$	4	$mm2, 4mm, 3m, 6mm$
$M\rho\alpha\eta\zeta$	4	$2'2'2, 42'2', 32', 62'2'$
$MP\alpha\eta\zeta$	4	$m'm'2, 4m'm', 3m', 6m'm'$
$M\alpha\eta\zeta$	2	$\bar{4}, \bar{4}2'm'$
$P\alpha\eta\zeta$	2	$4', 4'm'm$
$\rho\alpha\eta\zeta$	2	$\bar{4}', \bar{4}'m2'$
$\alpha\eta\zeta$	9	$222, 422, \bar{4}2m, 4'22', \bar{4}'2m', 32, 622, 23, 432$
$M\eta$	12	$\bar{1}, 2/m, 2'/m', m'm'm, 4/m, 4/mm'm', \bar{6}, \bar{3}, 6/m, \bar{3}m', \bar{6}m'2', 6/mm'm'$
$P\zeta$	12	$1', 21', m1', mm21', 41', 4mm1', 6', 31', 61', 3m1', 6'mm', 6mmm1'$
$\rho\alpha$	12	$\bar{1}', 2/m', 2'/m, mmm', 4/m', 4/m'mm, \bar{6}', \bar{3}', 6/m', \bar{3}'m, \bar{6}'m2', 6/m'mm$
η	10	$mmm, 4'/m, 4/mmm, 4'/mmm', \bar{3}m, \bar{6}m2, 6/mmm; m3, \bar{4}3m, m3m$
ζ	10	$2221', \bar{4}1', 42221', \bar{4}2m1', 321', 6'2'2, 62221', 231', 4'32', 4321'$
α	10	$mm'm', 4'/m', 4/m'm'm', 4'/m'm'm, \bar{3}'m', \bar{6}'m'2, 6/m'm'm, m'3, \bar{4}'3m', m'3m'$
ϕ	20	$\bar{1}1', 2/m1', mmm1', 4/m1', 4/mm1', \bar{3}1', 6/m1', \bar{3}m1', 6/mmm1', m31', m3m1', 6'/m, 6'/m'mm', \bar{6}m21', 6'/mmm', m3m', \bar{4}3m1', m'3m$
Total	122	

The meaning of the dimensionless tensors κ^B, λ and φ can be best understood when they are compared with the tensors occurring in the more familiar constitutive equations

$$P = \varepsilon_0 \kappa^H E + \frac{1}{c}\alpha H$$

$$M = \frac{1}{c}\tilde{\alpha} E + \mu_0 \chi^E H$$

Here χ^E is the magnetic susceptibility at constant E:

$$\chi^E = \mu^E - 1$$

then

$$\varphi^E = (\mu^E)^{-1} - 1 = -\chi^E(\mu^E)^{-1} = -(\mu^E)^{-1}\chi^E.$$

Furthermore κ^B, the electric susceptibility at constant B, is given by

$$\kappa^B = \kappa^H - \alpha\varphi^E\tilde{\alpha} = \kappa^H - \alpha\chi^E(\mu^E)^{-1}\tilde{\alpha}$$

and differs from that at constant H precisely when there is a magneto-electric effect ($\alpha \neq 0$). The magneto-electric coefficient λ for our choice of variables is related to the usual one by

$$\lambda = \alpha(\mu_E)^{-1}$$

After these identifications we find

$${}^1\mathscr{L} = \varepsilon_0[-\tfrac{1}{2}\kappa^B_{ik} E_i E_k - \lambda_{ik} E_i cB_k + \varphi^E_{ik} cB_i cB_k]$$

and we can verify that

$$-\frac{\partial^1 \mathscr{L}}{\partial E_i} = \varepsilon_0[\kappa^B_{ik} E_k + \lambda_{ik} cB_k] = {}^1P_i$$

$$-\frac{\partial^1 \mathscr{L}}{\partial B_i} = c\varepsilon_0[\lambda_{ki} E_k - \varphi^E_{ik} cB_k] = {}^1J_1$$

which corresponds indeed to

$$c\frac{\partial^1 \mathscr{L}}{\partial F_{\alpha\beta}} = {}^1P^{\alpha\beta} = \tfrac{1}{2}\chi^{\alpha\beta\sigma\delta}F_{\sigma\delta}$$

We are not yet finished with ${}^1\mathscr{L}$. We are now interested in moving crystals. Therefore we have to compute

$$c\frac{\partial^1 \mathscr{L}}{\partial F_{\alpha'\beta'}} = \tfrac{1}{2}\Lambda^{\alpha'}_{\alpha}\Lambda^{\beta'}_{\beta}\chi^{\alpha\beta\sigma\delta}F_{\sigma\delta} = {}^1P^{\alpha'\beta'}$$

where $\Lambda^{\alpha'}_{\alpha}$ is the matrix of a Lorentz transformation the inverse of which is denoted by $\Lambda^{\alpha}_{\alpha'}$. The result, in terms of the polarization ${}^1P$ and magnetization ${}^1M$ induced by the fields, has the well-known form

$$c^1\mathbf{P}_{\perp} = \frac{1}{\gamma}[c^1P + \beta \times {}^1J]_{\perp}, \quad c^1\mathbf{P}_{\parallel} = c^1P$$

$${}^1\mathbf{J}_{\perp} = \frac{1}{\gamma}[{}^1J - \beta \times c^1P]_{\perp}, \quad {}^1J_{\parallel} = {}^1J_{\parallel}$$

To simplify our writing, we now choose our coordinates so that the velocity of the crystal is parallel to the 2-direction. Thus

$$\Lambda_{\alpha}^{\alpha'} = \begin{pmatrix} \frac{1}{\gamma} & 0 & \frac{\beta}{\gamma} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\beta}{\gamma} & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_{\alpha'}^{\alpha} \Lambda_{\alpha}^{\alpha'} = g_{\beta}{}^{\alpha}$$

Then we have, for instance,

$$c^1 \mathbf{P}_1 = \frac{1}{\gamma} \frac{1}{R_0} \left\{ (\kappa_{1k}^B E_k + \lambda_{1k} cB_k) + \frac{v_2}{c} (\lambda_{k3} E_k + \varphi_{3k}^E cB_k) \right\}$$

Up to first order in the velocity, this gives

$$c\,^1\mathbf{P}_1 = c\,^1\mathbf{P}_1 + \varepsilon_0 (\lambda_{k3} - \tfrac{1}{2}\kappa_{1k}^B)\, v_2 E_k + \varepsilon_0 (\varphi_{3k}^E - \lambda_{1k})\, v_2\, cB_k$$

We see, therefore, that the Lorentz transformation gives rise to higher order kinetic effects. The ordinary kinetic effect arises (to first order) when there are spontaneous polarizations. These are taken care of by the second term, $^2\mathscr{L}$, of the Lagrangian density.

For this term we write

$$^2\mathscr{L}(F_{..}, u_.) = \tfrac{1}{2} \xi^{\sigma\alpha\beta} \mu_\sigma F_{\alpha\beta}$$

and we identify the tensor ξ in the following way

$$\xi^{0\alpha\beta} = \frac{\gamma}{c^2}\, {}^0p^{\alpha\beta} = \frac{\gamma}{c^2} \begin{pmatrix} 0 & -c\,^0P_1 & -c\,^0P_2 & -c\,^0P_3 \\ c\,^0P_1 & 0 & ^0J_3 & -^0J_2 \\ c\,^0P_2 & -^0J_3 & 0 & ^0J_1 \\ c\,^0P_3 & ^0J_2 & -^0J_1 & 0 \end{pmatrix}$$

(where $^0p^{\alpha\beta}$ is of course the tensor of spontaneous polarization). Furthermore

$$\xi^{i\alpha\beta} = \gamma \begin{pmatrix} 0 & \eta_{i1} & \eta_{i2} & \eta_{i3} \\ -\eta_{i1} & 0 & -\zeta_{i3} & \zeta_{i2} \\ -\eta_{i2} & \zeta_{i3} & 0 & -\zeta_{i1} \\ -\eta_{i3} & -\zeta_{i2} & \zeta_{i1} & 0 \end{pmatrix}$$

(where the η_{ik} and ζ_{ik} are the kinetic coefficients introduced above). The physical dimension of ξ is As^2/m^3, the same as that of the ratio of the mass m_e to the Bohr magneton b_e of the electron:

$$\frac{m_e}{b_e} = 0.785 \frac{As^2}{m^3}$$

It is interesting to note that the tensor depends on the symmetry of the crystal; for some groups it must vanish. These are the 42 groups listed under $p\alpha$, α and ϕ in Table III.

With this identification, $^2\mathscr{L}$ can be transcribed as

$$^2\mathscr{L} = -\{^0P \cdot E + {}^0J \cdot B + \eta_{ik} v_i E_k + \zeta_{ik} v_i cB_k\}$$

Again we compute the contribution of this term to the electromagnetic polarization of the moving crystal:

$$c \frac{\partial^2 \mathscr{L}}{\partial F_{\alpha'\beta'}} = {}^2\mathbf{P}^{\alpha'\beta'} = c \Lambda_{\alpha}^{\alpha'} \Lambda_{\beta}^{\beta'} \xi^{\sigma\alpha\beta} u_\sigma = \Lambda_{\alpha}^{\alpha'} \Lambda_{\beta}^{\beta'}\, {}^2p^{\alpha\beta}$$

Here we have introduced the abbreviation

$$^2p^{\alpha\beta} = c\xi^{\sigma\alpha\beta} u_\sigma$$

which is equivalent to

$$^2P = {}^0P + \eta v$$

$$^2J = {}^0J + c\zeta v$$

Then we find of course

$$c\,^2\mathbf{P}_\perp = \frac{1}{\gamma} [c\,^2P + \beta \times {}^2J]_\perp, \quad c\,^2\mathbf{P}_\parallel = c\,^2P$$

$$^2\mathbf{J}_\perp = \frac{1}{\gamma} [^2J - \beta \times c\,^2P]_\perp, \quad ^2\mathbf{J} = {}^2J$$

We can now confirm the existence of first-order kinetic effects due to the spontaneous polarizations, by considering the first-order development of the expressions above. For our choice of the velocity in the 2-direction, we find e.g.

$$c\,^2\mathbf{P}_1 = \frac{1}{\gamma} \left\{ c(^0P_1 + \eta_{21} v_2) + \frac{v_2}{c} (^2J_3 + c\zeta_{23} v_2) \right\}$$

$$= c\,^0P_1 + \left[c\eta_{21} + \frac{1}{c} (^0J_3 - \tfrac{1}{2} c\,^0P_1)\, v_2 \right]$$

This concludes our calculation of the polarization of a moving crystal when the symmetry permits spontaneous polarization and first-order kinetic effects:

$$c\mathbf{P}_\parallel = \left(c\,^0P + \frac{1}{R_0} \kappa^B E + \frac{1}{R_0} \lambda cB + \eta v \right)_\parallel$$

$$\mathbf{J}_\parallel = \left(^0J + \frac{1}{R_0} \tilde{\lambda} E - \frac{1}{R_0} \varphi^E cB + c\zeta v \right)_\parallel$$

$$c\mathbf{P}_\perp = \frac{1}{\gamma}\left[c\,{}^0P + \frac{1}{R_0}\kappa^B E + \frac{1}{R_0}\lambda cB + \eta v\right.$$

$$\left.\beta \times \left({}^0J + \frac{1}{R_0}\tilde{\lambda}E - \frac{1}{R_0}\varphi^E\, cB + c\zeta v\right)\right]_\perp$$

$$J_\perp = \frac{1}{\gamma}\left[\,{}^0J + \frac{1}{R_0}\tilde{\lambda}E - \frac{1}{R_0}\varphi^E cB + c\zeta v\right.$$

$$\left.\beta \times \left(c\,{}^0P + \frac{1}{R_0}\kappa^B E + \frac{1}{R_0}\lambda cB + \eta v\right)\right]_\perp$$

Indeed the third term, ${}^3\mathscr{L}$, of the Lagrangian density does not contribute to the polarization. It does contribute, however, to the momentum. We shall now discuss ${}^3\mathscr{L}$, then calculate the contributions of all three terms to the momentum of the moving crystal.

Now this third term ${}^3\mathscr{L}$ clearly must be related to the kinetic energy which was still missing in our Lagrangian density. It will also contain the ferrokinetic effect. Indeed we put

$${}^3\mathscr{L} = \tfrac{1}{2}\hat{\rho}^{\alpha\beta}\, u_\alpha u_\beta$$

and identify the components of $\hat{\rho}^{\alpha\beta}$ in the following way:

$$\hat{\rho}^{\alpha\beta} = \gamma_2 \begin{pmatrix} -{}^0\rho & \frac{1}{c}\,{}^0p_1 & \frac{1}{c}\,{}^0p_2 & \frac{1}{c}\,{}^0p_3 \\ \frac{1}{c}\,{}^0p_1 & -\rho_{11} & -\rho_{12} & -\rho_{13} \\ \frac{1}{c}\,{}^0p_2 & -\rho_{12} & -\rho_{22} & -\rho_{23} \\ \frac{1}{c}\,{}^0p_3 & -\rho_{13} & -\rho_{23} & -\rho_{33} \end{pmatrix}$$

Then ${}^3\mathscr{L}$ can be transcribed as

$${}^3\mathscr{L} = -(\tfrac{1}{2}\,{}^0\rho c^2 + {}^0p\cdot v + \tfrac{1}{2}\rho_{ik} v_i v_k)$$

The physical dimension of $\hat{\rho}$ is $\mathrm{VAs^3/m^5}$.

We now start determining the linear momentum of a crystal moving in a constant and uniform electromagnetic field.

The first contribution is

$$\begin{aligned} {}^1p^\mu &= -\frac{1}{\gamma}\frac{\partial\,{}^1\mathscr{L}}{\partial u_\mu} \\ &= -\frac{1}{\gamma}\frac{\partial}{\partial u_\mu}\left\{\frac{1}{8c}\Lambda^{\alpha'}_\alpha \Lambda^{\beta'}_\beta \Lambda^{\sigma'}_\sigma \Lambda^{\delta'}_\delta \chi^{\alpha\beta\sigma\delta} F_{\alpha'\beta'} F_{\sigma'\delta'}\right\} \\ &= -\frac{1}{\gamma c} K^{\mu\nu}_{\ \ \alpha} T_\nu^{\ \alpha} \end{aligned}$$

with

$$K^{\mu\nu}_{\ \ \alpha} := \frac{\partial \Lambda^{\alpha'}_\alpha}{\partial u_\mu}\Lambda^\nu_{\alpha'}$$

and

$$T_\nu^{\ \alpha} := \tfrac{1}{2}\chi^{\alpha\beta\sigma\delta} F_{\nu\beta} F_{\sigma\delta}$$

If you note that the tensor $T_\nu^{\ \alpha}$ is related to the energy-momentum tensor $\mathscr{T}_\nu^{\ \alpha}$ in the following way

$$\mathscr{T}_\nu^{\ \alpha} = \delta_\nu^{\ \alpha}\mathscr{L} - T_\nu^{\ \alpha}$$

the final result will not come as a surprise.

For our choice of the velocity in the 2-direction, we find

$${}^1\mathbf{p}^\mu = -\frac{1}{\gamma c}(K^{\mu 0}_{\ \ 2} T_0^{\ 2} + K^{\mu 2}_{\ \ 0} T_2^{\ 0})$$

Furthermore

$$K^{20}_{\ \ 2} = K^{02}_{\ \ 2} =: K^0 = -\frac{\gamma}{c\beta} = -\frac{1}{\beta}K^2$$

$$K^{00}_{\ \ 2} = K^{22}_{\ \ 0} =: K^2 = \frac{\gamma}{c} = -\beta K^0$$

and of course

$$K^{10}_{\ \ 2} = K^{12}_{\ \ 0} =: K^1 = 0$$

$$K^{30}_{\ \ 2} = K^{32}_{\ \ 0} =: K^3 = 0$$

Thus

$${}^1\mathbf{p}^0 = \frac{1}{\beta c^2}(T_0^{\ 2} + T_2^{\ 0}) = -\frac{1}{\beta}\,{}^1\mathbf{p}^2$$

$${}^1\mathbf{p}^2 = -\frac{1}{c^2}(T_0^{\ 2} + T_2^{\ 0}) = -\beta\,{}^1\mathbf{p}^0$$

Now it is easily verified that

$$T_0{}^2 = (E \times {}^1J)_2$$

and

$$T_2{}^0 = c^2({}^1P \times B)_2$$

where ${}^1P$, ${}^1J$ are well-known functions of the applied electromagnetic field, which we have already seen previously. Thus the final result is

$${}^1\mathbf{p}^0 = \frac{1}{\beta}\left(\frac{1}{c^2} E \times {}^1J + {}^1P \times B\right)_{\parallel}$$

$${}^1\mathbf{p}^2 = -\left(\frac{1}{c^2} E \times {}^1J + {}^1P \times B\right)_{\parallel} = -\beta^1 p^0$$

The second contributions is

$${}^2\mathbf{p}^\mu = -\frac{1}{\gamma}\frac{\partial^2 \mathscr{L}}{\partial u_\mu}$$

$$= -\frac{1}{\gamma}\tfrac{1}{2}\xi^{\mu\alpha\beta} F_{\alpha\beta} - \frac{1}{\gamma}\frac{\partial}{\partial u_\mu}\{\tfrac{1}{2}\Lambda_\sigma^{\sigma'}\Lambda_\alpha^{\alpha'}\Lambda_\beta^{\beta'}\xi^{\sigma\alpha\beta} u_{\sigma'} F_{\alpha'\beta'}\}$$

$$= {}^2p^\mu - \frac{1}{\gamma} K^{\mu\nu}_{\ \ \alpha} X_\nu^{\ \alpha}$$

where we have defined

$${}^2p^\mu = -\frac{1}{\gamma}\tfrac{1}{2}\xi^{\mu\alpha\beta} F_{\alpha\beta} = -\frac{1}{\gamma}\left(\frac{\partial^2 \mathscr{L}}{\partial u_\mu}\right)_{\Lambda=0}$$

and

$$X_\nu^{\ \alpha} = \tfrac{1}{2}\xi^{\alpha\sigma\beta} u_\nu F_{\sigma\beta} + \xi^{\sigma\alpha\beta} u_\sigma F_{\nu\beta}$$

It is perhaps worthwhile to single out

$${}^2p^0 = -\frac{1}{\gamma}\tfrac{1}{2}\xi^{0\alpha\beta} F_{\alpha\beta} = -\frac{1}{c^2}\tfrac{1}{2}{}^0P^{\alpha\beta} F_{\alpha\beta}$$

$$= \frac{1}{c}({}^0P\cdot E + {}^0J\cdot B)$$

In view of our choice of the velocity

$${}^2\mathbf{p}^\mu = {}^2p^\mu - \frac{1}{\gamma} K^\mu (X_0{}^2 + X_2{}^0)$$

The computation of the tensor X gives

$$X_0{}^2 = \tfrac{1}{2}\xi^{2\sigma\beta} u_0 F_{\sigma\beta} + \xi^{\sigma 2\beta} u_\sigma F_{0\beta}$$

$$= c(\eta_{2k} E_k + \zeta_{2k} cB_k) + \frac{1}{c}[({}^0J_1 + c\zeta_{21} v_2) E_3 - ({}^0J_3 + c\zeta_{23} v_2) E_1]$$

$$= \left(c\,{}^2p + \frac{1}{c} E \times {}^2J\right)_2$$

and

$$X_2{}^0 = \tfrac{1}{2}\xi^{0\sigma\beta} u_2 F_{\sigma\beta} + \xi^{\sigma 0\beta} u_\sigma F_{2\beta}$$

$$= -\frac{v_2}{c^2}\tfrac{1}{2}{}^0P^{\alpha\beta} F_{\sigma\beta} + c[({}^0P_3 + \eta_{23} v_2) B_1 - ({}^0P_1 + \eta_{21} v_2) B_3]$$

$$= \frac{v_2}{c}({}^0P\cdot E + {}^0J\cdot H) + (c\,{}^0P \times B)_2$$

$$= (-v\,{}^2p^0 + c^2 P \times B)_2$$

so that we finally find

$${}^2\mathbf{p}^0 = \frac{1}{\beta}\left({}^2p + \frac{1}{c^2} E \times {}^2J + {}^2P \times B\right)_{\parallel}$$

$${}^2\mathbf{p}^2 = \left(\frac{1}{c}\,{}^2p^0 v + \frac{1}{c^2} E \times {}^2J + {}^2P \times B\right)_{\parallel}$$

$${}^2\mathbf{p}^1 = {}^2p^1 = \eta_{1k} E_k + \zeta_{1k} cB_k$$

$${}^2\mathbf{p}^3 = {}^2p^3 = \eta_{3k} E_k + \zeta_{3k} cB_k$$

The third contribution to the linear momentum is

$${}^3\mathbf{p}^\mu = -\frac{1}{\gamma}\frac{\partial^3 \mathscr{L}}{\partial u_\mu}$$

$$= -\frac{1}{\gamma}\hat{\rho}^{\mu\beta} u_\beta - \frac{1}{\gamma}\frac{\partial}{\partial u_\mu}(\tfrac{1}{2}\Lambda_\alpha^{\alpha'}\Lambda_\beta^{\beta'}\hat{\rho}^{\alpha\beta} u_{\alpha'} u_{\beta'})$$

$$= {}^3p^\mu - \frac{1}{\gamma} K^\mu (\phi_0{}^2 + \phi_2{}^0)$$

where we have defined

$$^3p^\mu = -\frac{1}{\gamma}\left(\frac{\partial^3 \mathscr{L}}{\partial u_\mu}\right)_{\Lambda=0} = -\frac{1}{\gamma}\hat{\rho}^{\mu\beta} u_\beta$$

giving

$$^3p^0 = -\left(c\,{}^0\rho + \frac{1}{c}{}^0p\cdot v\right)$$

$$^3p_i = {}^0p_i + \rho_{ik} v_k$$

and

$$\phi_\nu{}^\alpha = \hat{\rho}^{\alpha\beta} u_\nu u_\beta$$

We find

$$\phi_0{}^2 = c\,{}^3p_2$$

$$\phi_2{}^0 = -v\,{}^3p^0$$

so that the final result, after some cancelling of terms, is

$$^3\mathbf{p}^0 = \frac{1}{\beta}{}^3p^2, \quad {}^3\mathbf{p}^1 = {}^3p^1$$

$$^3\mathbf{p}^2 = \beta\,{}^3p^0, \quad {}^3\mathbf{p}^3 = {}^3p^3$$

Adding the three contributions, we obtain as final result

$$\mathbf{p}_\parallel \equiv -\left\{\left[{}^0\rho + \frac{1}{c^2}({}^0P\cdot E + {}^0J\cdot B + {}^0p\cdot v)\right] v + \frac{1}{c^2}E\times\left({}^0J + \frac{1}{R_0}\tilde{\lambda}E - \frac{1}{R_0}\varphi^E cB + c\zeta v\right) + \left({}^0P + \frac{1}{R_0}\kappa^B E + \lambda cB + \eta v\right)\times B\right\}_\parallel$$

$$\mathbf{p}_\perp = ({}^0p + \eta E + \zeta cB + \rho v)_\perp$$

$$\mathbf{p}_0 = \frac{1}{|\beta|}\left\{({}^0p + \eta E + \zeta cB + \rho v) + \frac{1}{c^2}E \times\left({}^0J + \frac{1}{R_0}\tilde{\lambda}E - \frac{1}{R_0}\varphi^E cB + c\zeta v\right) + \left({}^0P \times \frac{1}{R_0}\kappa^B E + \frac{1}{R_0}\chi cB + \eta v\right)\times B\right\}_\parallel$$

Concerning the vector products that appear, let us note the following result:

$$\frac{1}{c^2}E\times J + P\times B = D\times B - \frac{1}{c^2}E\times H$$

so that this term corresponds to the Poynting vector of the moving polarized crystal where the contribution of the vacuum has been subtracted (and in vacuum this term vanishes). This corresponds exactly to the fact that we have started with a density of stored free enthalpy, i.e. a free enthalpy where the vacuum contributions have been subtracted.

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