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Relativistic Symmetries and Lower Bounds for the Magneto-Electric Susceptibility and the Ratio of Polarization to Magnetization in a Ferromagneto-Electric Crystal

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A definition of the relativistic symmetry of a polarized crystalline medium is proposed. The requirement of invariance of the susceptibility under this group gives rise to relations between the components of the tensor which then are used in the case of orthorhombic boracite to obtain lower bounds for the magneto-electric susceptibility and for the ratio of the spontaneous polarization to magnetization.

Es wird eine Definition der relativistischen Symmetrie eines polarisierten Kristalles vorgeschlagen. Die Forderung nach Invarianz des Suszeptibilitätstensors unter dieser Gruppe gibt Beziehungen zwischen den Komponenten des Tensors. Am Beispiel der orthorhombischen Borazite wird gezeigt, wie man daraus untere Schranken für die magnetoelektrische Suszeptibilität und das Verhältnis der spontanen elektrischen Polarisation zur spontanen Magnetisierung ableiten kann.

1. Ferromagneto-Electricity

Let $g(E, H, T)$ be the density of stored free enthalpy of a crystal. The crystal is ferroelectric if

$$\lim_{E\to 0} \frac{\partial g}{\partial E_i} =: {}^{\circ}P_i \neq 0 \,,$$

it is ferromagnetic if

$$\lim_{H\to 0} \frac{\partial g}{\partial H_i} =: {}^{\circ}M_i \neq 0 \,,$$

and it is ferromagneto-electric if both these conditions are fulfilled. If this is the case, then the symmetry of the crystal is necessarily also compatible with the magneto-electric effect:

$$\lim_{E\to 0,\, H\to 0} \frac{\partial^2 g}{\partial E_i\, \partial H_k} =: \alpha_{ik} \neq 0 \,,$$

but of course the magneto-electric effect may exist in absence of ferromagneto-electricity.

The question now arises whether there is a relation between the vectors ${}^{\circ}\boldsymbol{P}$ and ${}^{\circ}\boldsymbol{M}$, and the tensor $\boldsymbol{\alpha}$. The research described in this paper was motivated by the desire to see whether this would be the case in nickel–iodine–boracite $Ni_3B_7O_{13}I$ (in short NIB), the crystal in which ferromagneto-electricity was first discovered in 1966 [1].

In a general ferromagneto-electric crystal the stored free enthalpy and the constitutive equations for the polarization P_i and the magnetization M_i may be represented in the following way:

$$-g(E, H, T) = {}^{\circ}P_i(T)\, E^i + {}^{\circ}M_i(T)\, H^i + \frac{1}{2}\, \varepsilon_0 \varkappa_{ik}(T)\, E^i E^k +$$

$$+ \frac{1}{c}\, \alpha_{ik}\, (T)\, E^i H^k + \frac{1}{2}\, \mu_0 \mu_{ik}(T)\, H^i H^k + \cdots\,, \tag{1}$$

$$-\frac{\partial g}{\partial E^i} = P_i = {}^{\circ}P_i + \varepsilon_0 \varkappa_{ik} E^k + \frac{1}{c}\, \alpha_{ik}\, H^k + \cdots\,, \tag{2}$$

$$-\frac{\partial g}{\partial H^i} = M_i = {}^{\circ}M_i + \frac{1}{c}\, \alpha_{ki} E^k + \mu_0 \mu_{ik} H^k + \cdots\,. \tag{3}$$

Notice that rationalized MKS units are used in this paper ($\varepsilon_0 \mu_0 = 1/c^2$).

If only terms up to second order are kept in g, we speak of linear ferromagneto-electric crystals; if the spontaneous quantities ${}^{\circ}P_i$ and ${}^{\circ}M_i$ are also omitted, we have a linear magneto-electric substance. Dzyaloshinski [2] has introduced the constitutive equations for the latter case, O'Dell [3], however, seems to have been the first to realize that such equations may be written most conveniently when the antisymmetric tensors of electromagnetic field and polarization (as encountered in special relativity) are used. If one wants — as we do here — to explore and to exploit the possible relativistic symmetries of a polarized crystal, this way of representing thermodynamic potentials (1) and constitutive equations (2), (3) is indispensable. (First results have been announced in [4].) We shall use the following transcriptions:

$$g = \frac{1}{8c}\, \chi^{\alpha\beta\sigma\delta}\, F_{\alpha\beta} F_{\sigma\delta}\,,$$

$$c\, \frac{\partial g}{\partial F_{\alpha\beta}} =: J^{\alpha\beta} = J^{\alpha\beta} + \frac{1}{2}\, \chi^{\alpha\beta\sigma\delta}\, F_{\sigma\delta}\,.$$

The tensors $\mathbf{F}$ and $\mathbf{J}$ occurring here have the following components: $F_{i0} = E_i$, $F_{23} = cB_1$, $F_{31} = cB_2$, $F_{12} = cB_3$ and $J^{0i} = cP_i$, $J^{32} = J_1$, $J^{13} = J_2$, $J^{21} = J_3$. The magnetizations M and J are related by $M = \mu_0 J$. Representing pairs of indices by a single one ($01 = 1$, $02 = 2$, $03 = 3$, $23 = 4$, $31 = 5$, $12 = 6$), the relativistic susceptibility tensor can be presented as a six-by-six matrix:

$$\chi = \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2} \begin{pmatrix} -\varkappa^B & \lambda \\ \tilde{\lambda} & \Phi \end{pmatrix},$$

in terms of the three-by-three matrices Φ, $\varkappa^B$, λ. In a linear ferromagneto-electric, they are simply related to those of equations (2) and (3). One finds

$$\Phi = \mu^{-1} - 1 = -\chi \mu^{-1}\,.$$

Furthermore $\varkappa^B$, the electric susceptibility at constant B is given by

$$\varkappa^B = \varkappa - \alpha \mu^{-1} \tilde{\alpha}$$

and differs from that at constant H precisely when there is a magneto-electric effect ($\alpha \neq 0$). The magneto-electric coefficient is λ for our choice of variables and is related to the usual one, α, by

$$\lambda = \alpha \mu^{-1}\,.$$

2. About the Lorentz Group

The subgroup $K(J^{\alpha\beta})$ of the Lorentz group that leaves invariant a polarization tensor $J^{\alpha\beta}$ plays an important role in the dertermination of the relativistic symmetry group of a polarized crystalline medium as it is defined here. We shall give a simple derivation of K, sufficient for the case in which we are interested. (For another derivation, see [5].) First, however, we must characterize various types of elements of the Lorentz group 0(3, 1).

This group is a semidirect product of its component CO(3, 1), connected to the unity, with a dihedral group D_2 for which there is a choice of several realizations:

$$O(3,1) = CO(3,1) \dot{\times} D_2 \,,$$

$$D_2 = \bar{1}1', m1', 2'/m', 2'/m, mm'2' \,.$$

The dotted cross is used to denote a semidirect product. The Shubnikov point groups are finite subgroups of the group

$$O(3)1' = SO(3) \dot{\times} D_2 \,,$$

$$D_2 = 11', m1', 2'/m', 2'/m, mm'2' \,.$$

This is a subgroup of the Lorentz group that leaves invariant a four-dimensional positive-definite quadratic form. Special elements of the Lorentz group are the boosts in a space-direction $\boldsymbol{d}$, which we shall denote by $b_{\boldsymbol{d}}(\beta)$, $b_{\boldsymbol{d}}$ or b, where $\beta = v/c$. They are subgroups of the abelian Lorentz group $CO_{\boldsymbol{d}}(1, 1)$ of the $(\boldsymbol{d}, ct)$ plane. Other special elements are rotations around some space-direction $\boldsymbol{d}$. These elements are noted $\varrho_{\boldsymbol{d}}(\varphi)$, $\varrho_{\boldsymbol{d}}$ or ϱ and belong to the abelian rotation group $SO_{\boldsymbol{d}}(2)$ of the plane perpendicular to $\boldsymbol{d}$.

However we shall need finer distinctions. Recall, therefore, that a linear transformation in a n-dimensional vector space has n (affine) invariants $I_1, \ldots, I_n$, where I_1 is the trace and I_n the determinant of the transformation. If these transformations leave invariant n-dimensional quadratic form it may be shown that there exist the following relations between the invariants

$$I_n I_k = I_{n-k} \qquad (I_n = \pm 1) \,.$$

In the case of CO(3, 1) we have

$$I_3 = I_1 \,, \qquad I_4 = 1 \,,$$

so that the two invariants I_1 and I_2 may be used to characterize the elements of this group.

For any element of CO(3, 1), different from the identity we have either

$$I_1 = 4 \,, \qquad I_2 = 6$$

or

$$I_2 = 2(k + K) \,, \qquad I_2 = 2(2kK + 1) \,,$$

where

$$k = \cos\varphi \,, \qquad K = \cosh\chi \,.$$

This is an invariant formulation of a theorem due to Wigner, for which I know of no other reference than [6].

In the first case the elements are called parabolic (they are not diagonizable); in the second case they are called non-parabolic. Notice that parabolic elements have the same invariants as has the identity.

For non-parabolic elements I_1 is positive, while I_2 may take any value. There are two special cases of non-parabolic elements: the elliptic elements for which we have

$$I_1 = 2(k+1) \qquad 0 \leqq I_1 < 4\,,$$
$$I_2 = 2(2k+1) \qquad -2 \leqq I_2 < 6\,,$$

and the hyperbolic elements characterized by

$$I_1 = 2(K+1) \qquad 4 < I_1\,,$$
$$I_2 = 2(2K+1) \qquad 6 < I_2\,.$$

One sees that parabolic elements are at the limit between elliptic and hyperbolic elements. For these three types of element

$$2(I_1 - 1) - I_2 = 0$$

so that (in addition to I_4) they have in fact only a single invariant. Non-parabolic elements that are neither elliptic nor hyperbolic have

$$2(I_1 - 1) - I_2 = 4(1-k)\,(K-1) > 0\,.$$

All rotations are elliptic. The converse is not true; however, any elliptic element is conjugate in CO(3, 1) to a rotation. All boosts are hyperbolic. The converse is not true; however, any hyperbolic element is conjugate in CO(3, 1) to a boost.

3. The Symmetry of the Polarization Tensor

We shall now investigate the relativistic symmetry K of the tensor of spontaneous electromagnetic polarization and limit ourselves here to the case where the electric polarization is perpendicular to the magnetization. This occurs in the orthorhombic phase of NIB. Defining

$$a = c\frac{|\boldsymbol{P}|}{|\boldsymbol{J}|} = \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \frac{|\boldsymbol{P}|}{|\boldsymbol{M}|}$$

and choosing a convenient orientation of the coordinate system, we find for the spontaneous polarization tensor the following components $J^{03} = a|\boldsymbol{J}|$, $J^{32} = |\boldsymbol{J}|$. The Lorentz invariants are

$$\Phi_1 = |\boldsymbol{J}|^2 - c^2|\boldsymbol{P}|^2 = |\boldsymbol{J}|^2\,(1 - a^2)\,,$$
$$\Phi_2 = \boldsymbol{J}\cdot c\boldsymbol{P} = 0\,.$$

If $a = 1$, both invariants vanish. This case requires special consideration and will not be dealt with here (see however [5]).

If $a > 1$, then $\Phi_1 < 0$ and there exists a Lorentz transformation which transforms the magnetization to zero.

If $a < 1$, then $\Phi_1 > 0$ and there exists a Lorentz transformation which transforms the electric polarization to zero.

In these cases it is thus sufficient to find these "reducing" transformations and to determine the symmetry group of $\boldsymbol{P}$ or that of $\boldsymbol{J}$. It can easily be seen that in both cases we may choose as reducing transformation a boost $b_y(\beta)$ in the y-direction (i.e. perpendicular to both $\boldsymbol{P}$ and $\boldsymbol{J}$ and such that $\boldsymbol{P}$, $\boldsymbol{J}$, and β form a right-handed coordinate system). For the absolute value of the velocity, we find then

$$a > 1: \quad |\beta| = \frac{1}{a}, \qquad a < 1: \quad |\beta| = a\,.$$

The next step is to determine the symmetry $K(P_z)$ of P_z and that $K(J_x)$ of J_x. The result is [5]:

$$K(P_z) = [CO_z(1, 1) \times SO_z(2)] \dot{\times} D_2\,,$$
$$D_2 = m_x 1', m_y 1', m_x m_y' 2_z', m_x' m_y 2_z'\,,$$

and

$$K(J_x) = [CO_x(1, 1) \times SO_x(2)] \dot{\times} D_2\,,$$
$$D_2 = 2_y'/m', 2_z'/m', m_x m_y' 2_z', m_x 2_y' m_z'\,.$$

Notice that the intersection of these two groups is a finite group of order four:

$$K(P_z) \cap K(J_x) = m_x m_y' 2_z'\,.$$

The relativistic symmetry group $K_\perp$ of perpendicular polarization and magnetization is much larger. We find it as follows.

Let $a > 1$. The $b_y(1/a)$ annihilates the magnetization. The remaining electric polarization is left invariant by $K(P_z)$. Then $b_y^{-1}(1/a)$ restores the magnetization. Therefore the symmetry in this case is

$$K_\perp(a > 1) = b_y^{-1}\left(\frac{1}{a}\right) K(P_z)\, b_y\left(\frac{1}{a}\right)$$

or more explicitly

$$K_\perp(a > 1) = [L_{yz}(a, \beta) \times L_{xy}(a, \varphi)] \dot{\times} \left[m_x \times 1' b_y^2\left(\frac{1}{a}\right)\right].$$

Here

$$L_{yz}(a, \beta) = b_y^{-1}\left(\frac{1}{a}\right) CO_z(1, 1)\, b_y\left(\frac{1}{a}\right) \subset CO_{yz}(2, 1)$$

is an abelian group of hyperbolic transformations and

$$L_{xy}(a, \varphi) = b_y^{-1}\left(\frac{1}{a}\right) SO_z(2)\, b_y\left(\frac{1}{a}\right) \subset CO_{xy}(2, 1)$$

an abelian group of elliptic transformations. Furthermore

$$b_y^{-1}\left(\frac{1}{a}\right) 1' b_y\left(\frac{1}{a}\right) = 1' b_y^2\left(\frac{1}{a}\right) = 1' b_y\left(\frac{2a}{a^2+1}\right) = 1' b_y^2(a)$$

and

$$b_y^{-1} m_x b_y = m_x\,.$$

Similarly one finds, for $a < 1$, the direct product of a hyperbolic abelian subgroup of $CO_{xy}(2, 1)$ and an elliptic abelian subgroup of $CO_{yz}(2, 1)$:

$$K_\perp(a < 1) = b_y^{-1}(a)\, K_x(J_x)\, b_y(a)\,,$$
$$K_\perp(a < 1) = [L_{xy}(a, \beta) \times L_{yz}(a, \varphi)] \dot{\times} [m_x \to 1' b_y^2(a)]\,.$$

Note that $m_x m_y' 2_z'$ is contained in both $K_\perp(a > 1)$ and $K_\perp(a < 1)$.

In the exceptional case $a = 1$ we would find the direct product of two parabolic abelian subgroups of respectively $CO_{xy}(2, 1)$ and $CO_{yz}(2, 1)$.

4. The Role of the Normalizer in the Determination of the Relativistic Symmetry of a Crystal

Before explaining how to determine the relativistic symmetry of a crystal, let us dwell at some length on a simple illustration, which sheds light on the method that will be used.

Let us take a rectangle in the plane, say perpendicular to the z-axis. The elements that leave the rectangle invariant are easily found; they form the group

$$G = 2_z m_x m_y = \{1, 2_z, m_x, m_y\} \subset O_z(2) .$$

Notice that there is another rectangle of the same size that has the same symmetry group (Fig. 1). Now we want to know the elements of O(3) that leave the rectangle invariant. By inspection we find the group

$$G' = m_x m_y m_z = \{1, 2_z, 2_x, 2_y, \bar{1}, m_x, m_y, m_z\} \subset O(3) .$$

But suppose now that we have no intuition of 3-dimensional space, so that we would not have found the group G′ by inspection. How could we proceed to find nevertheless the three-dimensional symmetry of our two-dimensional figure?

Consider the normalizer

$$N := \{\boldsymbol{\alpha} \in O(3) | \boldsymbol{\alpha} G \boldsymbol{\alpha}^{-1} = G\}$$

of the original symmetry G in the larger group of transformations O(3); it is the subgroup of elements of O(3) that leave the group G globally (i.e. not necessarily elementwise) invariant; it is, so to speak, the symmetry of the symmetry. This group is easily found to be $N = 4_z/mmm$. Now a symmetry of the symmetry may play an important role, but it is not necessarily a symmetry of the object. As symmetry of the rectangle the normalizer is certainly too large a group, because it contains transformations of the plane that do not leave the rectangle invariant. If we consider the subgroups of the normalizer (Fig. 2), we see that these elements are contained in the subgroup 4mm of the normalizer. We certainly have to take as three-dimensional symmetry G′ of our object only a subgroup of the normalizer, and exactly a subgroup such that the elements of $O_z(2)$ that it contains form the original symmetry $G = 2_z m_x m_y$:

$$G' \subset N, \; G' \cap O_z(2) = G .$$

Such a group is not necessarily uniquely determined. In our example we find two groups satisfying this condition, viz.: $m_x m_y m_z$ and $\bar{4}_z m_x 2_e$. The first one,

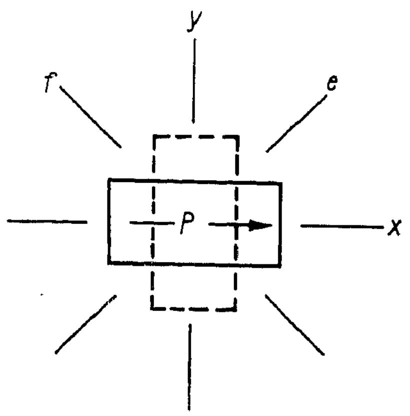

Fig. 1. Rectangles and axes of the plane

as we know, is the three-dimensional symmetry of our two-dimensional object. This can be easily seen by inspection, in Fig. 2, of the lattice of subgroups of 4_z/mmm. There subgroups of $O_z(2)$ have been underlined. They form of course a sublattice.

Let us try to learn something from the indeterminacy that we have found here. Let us first ask ourselves which three-dimensional objects have the above-mentioned groups as symmetry groups. The first group, mmm, describes a rectangular bipyramid, the second one $\bar{4}$m2 is the symmetry group of an object formed of a rectangular prism or pyramid in the upper half-space and another rectangular prism or pyramid in the lower half-space, the lower being turned by $\pi/2$ degrees with respect to the upper. Let us now transpose these findings to the situation that we shall consider later and take the pair-three-space and space-time instead of the pair two-space and three-space. We arrive at the following interpretation. The first group would describe an object that does not undergo any change at the time of observation $t = 0$. The second group describes an object that does undergo a drastic change at the time $t = 0$. In our example it turns by $\pi/2$ degrees. This change corresponds for instance to the switching from one of the twins in a crystal to the other. Except for this switching, the behaviour in the future and in the past of both objects is the same.

Let us remark that we would have found the same type of result starting with any regular n-gone in the plane. Only for $n = \infty$, i.e. for the circle do, the two kinds of behaviour in the added dimension disappear.

A second question is: why do we find two symmetries. The reason for this is obviously the fact that the rectangle in the plane is not uniquely determined by its symmetry; there are two rectangles of the same size having the same symmetry. The symmetry here is an incomplete description of the rectangle; it represents incomplete knowledge. Some information is missing; initial conditions or something else.

Since we are interested in polarized crystals, let this additional information be: the rectangle is electrically polarized in the positive x-direction. As is easily

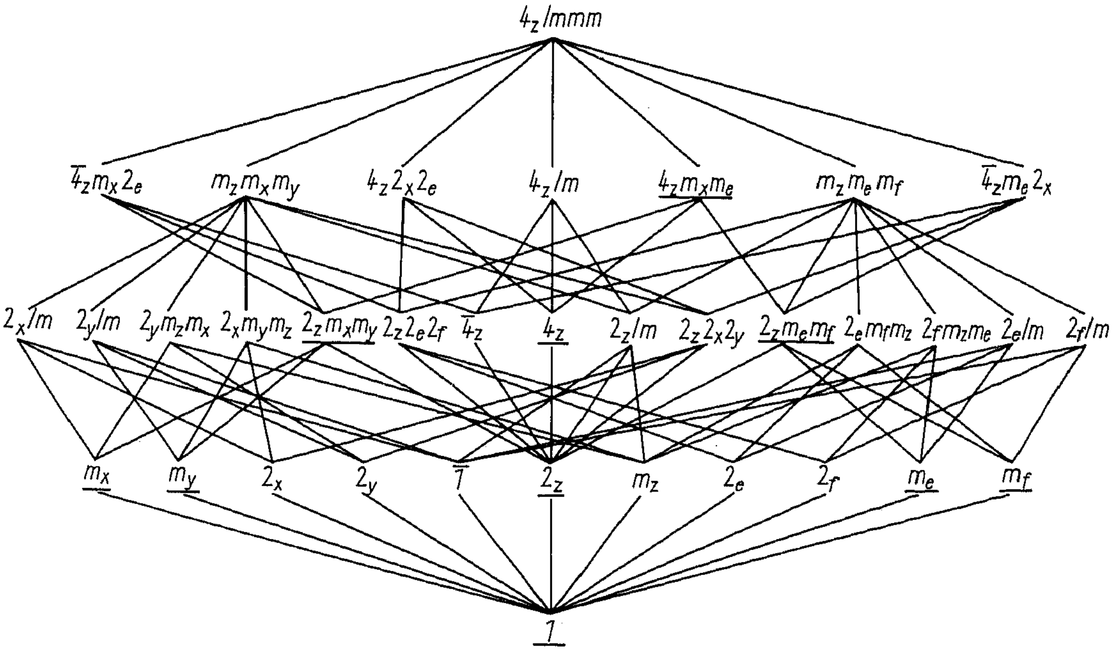

Fig. 2. Lattice of subgroups of 4_z/mmm, underlined: subgroups of $O_z(2)$

seen, its plane symmetry is now reduced to the group $G = m_y \subset O_z(2)$, whereas the symmetry with respect to O(3) is $G' = 2_x m_y m_z \subset O(3)$. Notice that the orientation is not the same as that of the group 2mm encountered previously. Let us now again proceed via the normalizer. The normalizer of m_y in O(3) is $N := N[m_y \subset O(3)] = \infty_y/mmm \subset O(3)$. But now we must take the polarization into account. Therefore we have to intersect the normalizer with the group of symmetry $K(P)$ in O(3) of the polarization. In this intersection we shall find the three-dimensional symmetry G′ of the polarized rectangle:

$$G' \subset N \cap K(P),\ G' \cap O_z(2) = G\,.$$

The symmetry of the polarization is $K(\boldsymbol{P}) = \infty_x\,mm \subset O(3)$ and the intersection with the normalizer is

$$\infty_y/mmm \cap \infty_x\,mm = 2_x m_y m_z\,.$$

But this is already the three-dimensional symmetry G′ we are looking for. Thus the answer is unique because the object was uniquely determined by the supplementary prescription that the rectangle be polarized in the x-direction; rectangle turned by $\pi/2$ would indeed be polarized in the y-direction and therefore excluded. This corresponds to the well-known fact that an applied field may be used to select one from among the possible domains of a ferro-electric crystal. Therefore now the switching — which we have found previously — from one domain to the other cannot take place. Notice that the circumstance of the intersection giving already the symmetry is not a generic property but pertains specifically to this example. The following, however, is a generic consequence of the procedure of finding the symmetry group G′ in higher dimensions as a subgroup of the normalizer of its symmetry group G in lower dimensions: the latter will always be a normal subgroup of the former.

We are now prepared to approach the determination of the relativistic symmetry group of a polarized crystal. We are given the Shubnikov point group G of a polarized crystal $G \subset O(3)1'$. We shall determine its normalizer in the Lorentz-group $N := N[G \subset O(3, 1)]$, intersect it with the relativistic group of the polarization $K(P^{\alpha\beta})$ and find the relativistic symmetry G′ of the polarized crystal as a subgroup of this intersection fulfilling the condition

$$G' \subset N \cap K(P^{\alpha\beta}),\ G' \cap O(3)1' = G\,.$$

5. The Structure of the Normalizer

Let us now investigate the structure of the normalizer of a Shubnikov point group in the Lorentz group.

First, recall that we have to distinguish four kinds of vector with respect to O(3)1′. Indeed consider the dihedral group of order four generated by the operations of space-inversion $\bar{1}$ and time-reversal 1′. Corresponding to the four irreducible representations of that group, one finds four types of vector, as shown in Table 1, which we call s-type, t-type, r-type and v-type vectors.

The following proposition can be easily proved.

Proposition: Let N be the normalizer of a Shubnikov point group G in O(3, 1) and let Ns be the normalizer of the same group in O(3) 1′: $Ns = N \cap O(3)\,1'$. Then $N \neq Ns$ if and only if the group G leaves a non-zero vector of type v invariant.

In consequence we have the following classification; the notations are the same as in the preceding proposition.

Table 1
With respect to O(3)1′ there are four types of vector:

$\bar{1}1'$	1	$\bar{1}$	1′	$\bar{1}'$		
Γ_s	1	1	1	1		$\dot{M}_i$
Γ_t	1	1	−1	−1	t	M_i
Γ_r	1	−1	1	−1	x_i	P_i
Γ_v	1	−1	−1	1	v_i	j_i

Classification: If G leaves no non-zero velocity invariant, then

$$N = N^s .$$

If the non-zero velocities left invariant by G are parallel to a single direction, say y, then

$$N = CO_y(1, 1) \dot{\times} N^s .$$

If the non-zero velocities left invariant by G may lie in a plane, say the xz-plane, then

$$N = O_{xz}(2, 1) \times m_y .$$

If finally the non-zero velocities left invariant by G may point in any direction then

$$N = O(3, 1) .$$

In Table 2 are listed the normalizers N^s and N for the 31 Shubnikov groups that leave a non-zero vector of type v invariant.

Table 2
Normalizers N^s and N of Shubnikov groups leaving v-type vectors invariant

G	N^s	N
mm′2′	2/mmm1′	CO(1,1)$\dot{\times}$ 2/mmm1′ = O(1,1) × 2mm
2mm, 22′2′, m′mm, $\bar{4}$′m2′, 4/m′mm	4/mmm1′	CO(1,1)$\dot{\times}$ 4/mmm1′ = O(1,1) × 4mm
3m, 32′, $\bar{3}$′m, $\bar{6}$′m2′	6/mmm1′	CO(1,1)$\dot{\times}$ 6/mmm1′ = O(1,1) × 6mm
4mm, 42′2′	8/mmm1′	CO(1,1)$\dot{\times}$ 8/mmm1′ = O(1,1) × 8mm
6mm, 62′2′, $\bar{6}$/m′mm	12/mmm1′	CO(1,1)$\dot{\times}$12/mmm1′ = O(1,1) × 12mm
2, 4, 3, 6, m′, 2/m′, $\bar{4}$′, 4/m′, $\bar{3}$′, $\bar{6}$′, 6/m′	∞/mmm1′	CO(1,1)$\dot{\times}$∞/mmm1′ = O(1,1) × O(2)
m, 2′, 2′/m	∞/mmm1′	O(2,1)$\dot{\times}$m$_\perp$
1, $\bar{1}$′	O(3)1′	O(3,1)

6. Application to Ferromagneto-Electric Boracite

Let us now return to our original motivation. Let us use the relativistic symmetry to find relations between the elements of the permeability tensor and those of the tensor of spontaneous polarization. We shall do this only for the case of orthorhombic NIB.

The crystal is ferromagneto-electric. If we choose the coordinate axes so as to have the spontaneous electric polarization $\boldsymbol{P}$ in the z-direction and the spontaneous magnetization $\boldsymbol{J}$ in the x-direction, then the invariant velocity points in the y-direction: J_x, v_y, P_z. The symmetry, then, is given by the Shubnikov group

$$\mathrm{G} = \mathrm{m}_x \mathrm{m}_y' 2_z' = \{1, \mathrm{m}_x, \mathrm{m}_y', 2_z'\} .$$

The normalizer of G in O(3, 1) is

$$\mathrm{N} = \mathrm{CO}_y(1, 1) \dot{\times} \mathrm{mmm}1' .$$

Unfortunately, there are no reliable data on the magnitude of the spontaneous electric polarization and magnetization in NIB. Taking, however, approximate data obtained from various samples, one finds

$$a = \frac{c|\boldsymbol{P}|}{|\mathbf{J}|} \approx 200 > 1 .$$

Thus the symmetry of the tensor $J^{\alpha\beta}$ of spontaneous polarization is

$$\mathrm{K}(J^{\alpha\beta}), = \mathrm{K}_\perp(a > 1) = [\mathrm{L}_{yz}(a, \beta) \times \mathrm{L}_{xy}(a, \beta)] \dot{\times} [\mathrm{m}_x \times 1' b_y^2(a)] .$$

The intersection of the normalizer with the group of the spontaneous polarization is

$$\mathrm{N} \cap \mathrm{K} = 1' b_y^2(a) \times \mathrm{mm}'2' .$$

This intersection does not contain any other elements of O(3) 1′ than those of the original symmetry group mm′2′. Therefore it is the relativistic symmetry group of the polarized crystal:

$$\mathrm{G}' = 1' b_y^2(a) \times \mathrm{mm}'2' .$$

Remember now

$$b^2(a) = b\left(\frac{2a}{a^2+1}\right).$$

In our case

$$\frac{2a}{a^2+1} \approx \frac{2}{a} = 2 \times 10^{-2} ,$$

i.e. the velocity corresponding to the boost is about a fiftieth of the velocity of light.

We now require that the permeability tensor $\chi^{\alpha\beta\sigma\delta}$ be invariant under the relativistic symmetry group G′.

Taking into account the Shubnikov symmetry $\mathrm{m}_x \mathrm{m}_y' 2_z'$, the susceptibility tensor

$$\boldsymbol{\chi}^{\alpha\beta\sigma\delta} = \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2} \begin{pmatrix} -\varkappa^B & \lambda \\ \lambda & \varphi \end{pmatrix}$$

has the following form:

$$\varkappa^B = \begin{bmatrix} \varkappa_{11} - \dfrac{\alpha_{13}^2}{\mu_{33}} & 0 & 0 \\ 0 & -\varkappa_{22} & 0 \\ 0 & 0 & \varkappa_{33} - \dfrac{\alpha_{31}^2}{\mu_{11}} \end{bmatrix} ,$$

$$\varphi = \begin{pmatrix} -\frac{\chi_{11}}{\mu_{11}} & 0 & 0 \\ 0 & -\frac{\chi_{22}}{\mu_{22}} & 0 \\ 0 & 0 & -\frac{\chi_{33}}{\mu_{33}} \end{pmatrix}, \qquad \lambda = \begin{pmatrix} 0 & 0 & \frac{\alpha_{13}}{\mu_{33}} \\ 0 & 0 & 0 \\ \frac{\alpha_{31}}{\mu_{11}} & 0 & 0 \end{pmatrix}.$$

Thus to implement the invariance with respect to G′, we have to require only the invariance with respect to

$$1'b^2\left(\frac{1}{a}\right) = \begin{pmatrix} -\frac{a^2+1}{a^2-1} & 0 & -\frac{2a}{a^2-1} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2a}{a^2+1} & 0 & \frac{a^2+1}{a^2-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The resulting two conditions are:

$$a(\varphi_{33} - \varkappa_{11}^{B}) = (a^2+1)\,\lambda_{13}\,,$$

$$a(\varphi_{11} - \varkappa_{33}^{B}) = -(a^2+1)\,\lambda_{31}\,,$$

or

$$\alpha_{13}^2 - \alpha_{13}\frac{a^2+1}{\alpha} = \varkappa_{11} + \varkappa_{33} + \varkappa_{11}\chi_{33}\,,$$

$$\alpha_{31}^2 + \alpha_{31}\frac{a^2+1}{a} = \varkappa_{33} + \chi_{11} + \varkappa_{33}\chi_{11}\,.$$

Using now the well known thermodynamic upper bounds for $\boldsymbol{\alpha}$ [7], [8]:

$$\varkappa_{11}\chi_{33} > \alpha_{13}^2\,, \qquad \varkappa_{33}\chi_{11} > \alpha_{31}^2\,,$$

we obtain

$$-\alpha_{13}\frac{a^2+1}{a} > \varkappa_{11} + \chi_{33}\,,$$

$$\alpha_{31}\frac{a^2+1}{a} > \varkappa_{33} + \chi_{11}\,.$$

This shows:

$$\alpha_{13} < 0,\ \alpha_{31} > 0.$$

Finally the inequalities limiting the possible values of $\boldsymbol{\alpha}$ may be cast in the following form:

$$(\varkappa_{11}\chi_{33})^{1/2} > |\alpha_{13}| > \frac{a}{a^2+1}(\varkappa_{11} + \chi_{33})\,,$$

$$(\varkappa_{33}\chi_{11})^{1/2} > \alpha_{31} > \frac{a}{a^2+1}(\varkappa_{33} + \chi_{11})\,.$$

These relations imply also lower bounds of the ratio a in terms of the electric and the magnetic susceptibility, namely:

$$\frac{a^2+1}{a} > \frac{\varkappa_{11}+\chi_{33}}{(\varkappa_{11}\chi_{33})^{1/2}}, \qquad \frac{a^2+1}{a} > \frac{\varkappa_{33}+\chi_{11}}{(\varkappa_{33}\chi_{11})^{1/2}}.$$

We have thus shown that our definition of relativistic symmetry of a polarized crystalline medium enables lower bounds to be deduced for the material coefficients $\boldsymbol{\alpha}$ and a.

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