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Permutation representations, epikernels and phase transitions

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Abstract. The subgroups that arise in phase transitions from a high-symmetry phase are characterized as those subgroups that are maximal with respect to the property of acting trivially on a given non-zero subspace U_i of the representation space M_i of a given irreducible representation T_i of H. In the case of subgroups of finite index the problem is reduced to that of studying faithful irreducible representations of finite groups. The use of permutation representations considerably simplifies the theory. Tables of the equitranslation epikernels of the space groups are given.

1. Introduction

In the previous paper (Ascher and Kobayashi 1977), we used permutation representations to solve the 'inverse Landau problem', namely to find, for a phase transition between two phases of symmetries H and $L < H$, the irreducible representation T_i of H that (according to Landau's theory) determines the transition. The group G determined by the transition is $G = H/K$, where K is the core of L (the kernel of the permutation representation of H on the co-sets of L in H). In all examples considered, we found the following: (i) G is a finite group that has one faithful irreducible representation τ_i; the representation T_i determined by the transition is then given by $T_i = \tau_i \circ \pi$ where π is the canonical epimorphism $H \twoheadrightarrow G$; and (ii) L is *maximal* with respect to the property that it acts trivially on *some* non-zero subspace of the representation space M_i of T_i (and τ_i).

Since permutation representations arise in such a natural way and so conspicuously simplify the solution of the inverse Landau problem, we shall base our group theoretic solution of the first part of the Landau problem on these representations. We shall indicate a procedure permitting us to find, for a given phase with symmetry group H, the possible phases with symmetries $L < H$. In this paper we limit ourselves to the case of *subgroups L of finite index n* in H.

The number of substructures (i.e. domains and/or sublattices) of symmetry L arising in such a transition is then n, so that there is a one-to-one correspondence between substructures and co-sets of L in H (Ascher 1971). Two of the interesting properties of permutation representations of a group on the co-sets of a subgroup is (i) that it is transitive, and (ii) that any transitive permutation representation of a group on a set of elements is isomorphic (as permutation representation) to the representation on the co-sets of the subgroup fixing one element of the set. Thus the permutation representations on the co-sets of L is, up to isomorphism, the representation that is transitive on the substructures arising from the transition.

2. Permutation representations

Let H be a (not necessarily finite) group and L a subgroup of finite index n:

$$L < H,\ H{:}L \ =:n < \infty. \tag{1}$$

Consider the decomposition into co-sets

$$H = \bigcup_{i=1}^{n} r_i L. \tag{2}$$

Then

$$hr_i \in r_{(\pi_L h)i} L \tag{3}$$

defines a permutation of co-sets (which does not depend on the choice of co-set representatives r_i) and hence a permutation representation π_L:

$$\pi_L{:}H \to \mathrm{S}_n. \tag{4}$$

Let K be the kernel of this representation (also called the core of L):

$$K{:}= \mathrm{Ker}\ \pi_L, \qquad K \lhd H, \qquad K \lhd L < H. \tag{5}$$

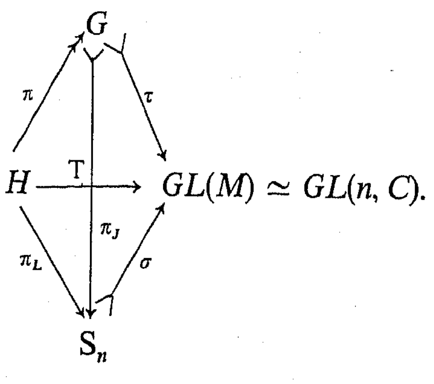
The group S_n is finite and the image $\mathrm{Im}\pi_L$ of H by π_L is a subgroup of S_n. Furthermore

$$\mathrm{Im}\tau_L \simeq H/K =:G. \tag{6}$$

Hence the index m of K in H is finite. Now, for any homomorphism $T{:}H \to H'$, there is a unique monomorphism $\tau{:}H/\mathrm{Ker}\ T \rightarrowtail H'$ such that

$$T = \tau \circ \pi,$$

where $\pi{:}H \twoheadrightarrow H/\mathrm{Ker}\ T$, the projection onto the quotient group. We therefore are in the position to establish the following diagram (in the category of groups):



(7)

The monomorphism σ associates with each permutation in S_n a 'permutation matrix' in $GL(n, C)$. This monomorphism assigns a complex (linear) representation T to the permutation representation π_L and also a faithful complex (linear) representation τ to the permutation representation π_J of G (where $\pi L \simeq L/K =:J < G$). We shall call T and τ the linear equivalents of π_L and π_J, respectively. Diagram (7) shows that, in the case of groups $L < H$ of finite index, problems concerning representations π_L of infinite groups H can be translated into problems concerning faithful representations of the finite group G. We shall see presently that the solutions to these problems can be transferred back to H.

Let us now consider the decomposition

$$\tau = \bigoplus_{i=l}^{t} n_i \tau_i \tag{8}$$

of the faithful representation τ of the finite group G into irreducible representations $\tau_i : G \to GL(M_i)$. We know that the linear equivalent $\tau = \sigma O \pi_J$ of the permutation representation π_J is in fact the representation of G induced by the unit representation I_J of the subgroup J:

$$\tau = \mathrm{Ind}^G I_J = : I_J^G. \tag{9}$$

We shall write 1_J^G for the character of I_J^G. Thus by Frobenius' reciprocity theorem

$$\langle 1_J^G | 1_G \rangle_G = \langle 1_J | 1_J \rangle_J = 1 \tag{10}$$

where

$$\langle \chi_1 | \chi_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_1^*(g) \chi_2(g) \tag{11}$$

for any two (complex-valued) class functions, χ_1 and χ_2, on G. This shows that τ contains the unit representation 1_G exactly once. Hence, (i) τ is always reducible (for $J \neq G$), and (ii) π is transitive on the co-sets of J in G.

More generally we have (writing χ_i for the character of τ_i):

$$\langle 1_J^G | \chi_i \rangle_G = \langle 1_J | \mathrm{Res}\, \chi_i \rangle_J = n_i \leqslant \dim M_i, \tag{12}$$

showing that n_i is the dimension of the subspace $U_i \subset M_i$ on which J (via τ) acts trivally. Res τ_i in (12) denotes the restriction of $r_i : G \to GL(M_i)$ to $J < G$. Restriction is also called subduction.

All these results can be lifted from (the finite group) G to H, because of the following (almost trivial) proposition. Let M be a representation space for H (i.e. an H-module) such that the action $T : H \to GL(M)$ has a kernel K of finite index in H. Then M is completely reducible (i.e. for every H-submodule N of M, there exists a H-submodule $\bar{N}$ of M such that $M = N \oplus \bar{N}$).

It follows that to the decomposition (8) there corresponds a decomposition

$$T = \bigoplus_{i=1}^{t} n_i T_i \tag{13}$$

with (possibly after renumbering)

$$T_i = \tau_i O \pi. \tag{14}$$

Furthermore, n_i, as given by (12), is the dimension of the subspace $U_i \subset M_i$ on which L (via T) acts trivially.

3. The Curie principle

According to Curie (1894), 'the characteristic symmetry of a phenomenon is the maximal symmetry compatible with the phenomenon. A phenomenon can exist in a medium which has its symmetry or that of one of the intergroups of the characteristic symmetry'. Curie terms intergroup what today we call subgroup. He furthermore says: 'If several phenomena are superposed.. ther remain as symmetry elements... only those that all phenomena have in common'.

Zhëludev and Shuvalov (1956) and subsequently Sonin and Zhëludev (1959) have applied Curie's principle to ferroelectric phase transitions (i.e. to transitions where the electrical polarization P of the crystal is the order parameter), taking crystallographic point groups and space groups, respectively, as symmetry groups H of the disordered phase. The symmetry group L of an ordered phase is then the intersection $H \wedge \Gamma$ of H and the symmetry group Γ of the polarization. It then follows that L is the largest subgroup of Γ that leaves the crystal invariant and also the largest subgroup of H that leaves the order parameter P (in a given orientation) invariant.

This latter property can be used to give the Curie principle an interpretation in terms of the group H only (Ascher 1966a, b). In the lattice of subgroups of H, one selects those that leave the order parameter Ω invariant (Ω-groups) and then selects those that are maximal with respect to *a given* Ω. In all the ferroelectric transitions considered by Ascher (1966a), the groups arising obey a stronger principle (termed *maximality principle*), and are maximal with respect to the property of leaving *some* orientation of the order parameter invariant. The situation is similar in the case of the more general transitions studied in the previous paper (Ascher and Kobayashi 1977).

4. Epikernels

By analogy with the Curie principle, we propose here as groups that can possibly arise in phase transitions from a phase with symmetry H only those $L < H$ that are *maximal* with respect to the property of acting trivially on a *given* (non-zero) subspace U_i of the representation space M_i of a *given* irreducible representation T_i of H. Equivalently this means that L is *maximal* with respect to the property that the linear equivalent I_L^H of the permutation representation π_L contains a *given* irreducible representation T_i of H a *given* number of times. The subgroups fulfilling these conditions will be termed *epikernels* of T_i and we write

$$L \in \mathrm{Ek}\ \mathrm{T}_i. \tag{16}$$

In more detail, (16) means that

$$\langle 1_L^H | \mathrm{T}_i \rangle = n_i \neq 0, \tag{16a}$$

and

$$\langle 1_F^H | \mathrm{T}_i \rangle = n_i, \qquad L \leqslant F \tag{16b}$$

imply

$$L = F. \tag{16c}$$

Epikernels were introduced by Melvin (1956) in a different context and in a different way under the name of co-kernels. The reasons for our change of denomination is that 'co-kernel' has by now a precise, and quite different, meaning in algebra. It is likely that this algebraic notion of co-kernel will play a role in the subsequent development of our theory. Epikernels (i.e. Melvin's co-kernels), however, are pull-backs (see e.g. MacLane and Birkhoff 1967). The epikernels of the crystallographic point groups have been determined by McDowell (1965). During the writing of the final version of this paper we became acquainted with the work of Murray-Rust *et al* (1975) and Janovec *et al* (1975) in which the epikernels of the crystallographic point groups have been determined and used. In the latter paper they appear in connexion with Birman's subduction criterion (Birman 1966). From Frobenius' reciprocity relation (12) one can see that the subduction criterion amounts to the following requirement: the irreducible representation $T_i : H \to GL(M_i)$ of H and the subgroup L of H have the property that L acts trivally on some non-trivial subspace of M_i. Among such subgroups L, Janovec *et al* admit only 'the maximal subgroups of those in which the same identity representations are subduced'. These are exactly the epikernels. A more restrictive selection is introduced below.

In this paper the relations between permutation representations and epikernels have been worked out for the first time. Also, the equitranslation epikernels of the crystallographic *space groups* have been tabulated. These are the symmetries of the homogeneous low-symmetry phases arising in phase transitions in which the number of atoms in the *primitive* unit cell does not change, The question of non-equitranslation epikernels and 'non-homogeneous' phases will te taken up in subsequent papers. The discussion of the relation of epikernels to subgroups admitted by the Curie principle also necessitates a separate paper; the results in both cases are almost the same.

The (unique) *minimal epikernel* of T_i is the kernel $K_i = \text{Ker}\, T_i$. Indeed the linear equivalent $I^H_{K_i}$ of the permutation representation π_{K_i} of H contains the irreducible representation T_i a maximal number of times, namely $d_i = \deg T_i = \dim M_i$ times. This can be realized by noticing that I_1^{H/K_i} is the regular representation of H/K_i. Equivalently this means of course that K_i is the epikernel of T_i which is trivial on the largest subspace of M_i, namely on M_i itself. It follows that

$$L \in \text{Ek}\, T_i \qquad \text{implies} \qquad \text{Ker}\, T_i = : K_i \leqslant L; \tag{17}$$

hence the name epikernel. Indeed, suppose that K_i is not contained in L. The the subspace $U_i \subset M_i$, fixed (element-wise) by L is also fixed (element-wise) by the subgroup $K_i L < H$. But $L < K_i L$, and this contradicts the hypothesis that L is maximal with respect to the property of fixing U_i. As a consequence of this we see that the trivial group 1 can arise in a phase transition from a group H if and only if H has faithful irreducible representations, whereas the Curie principle imposes no such restriction.

There are in general several *maximal epikernels* of a given irreducible representation T_i. They are also *maximal* with respect to the property that the linear equivalent I^H_L of the permutation representation π_L contains a *given irreducible representation* T_i of H. Equivalently these are those groups that are maximal with respect to the property that there *exists some non-zero subspace* U_i of the representation space M_i of T_i on which L (via T_i) acts trivially. The maximal epikernels are exactly the subgroups selected by the *maximality principle*. It seems (Ascher 1966a, b) that the maximal epikernels give those symmetries that arise from the minimization of the thermodynamic potential.

The necessary condition (17) for a group L to be an epikernel of the irreducible representation T_i of H can be put in other more useful forms. First we show that

$$I^H_L = \bigoplus_{j=1}^{s} n_j T_j \tag{18}$$

and

$$L \in \mathrm{Ek}\, \mathrm{T}_i \qquad i \in \{1, \ldots, s\} \tag{19}$$

implies

$$K = \mathrm{Ker}\, I_L^H = \mathrm{Ker}\, \mathrm{T}_i = K_i.$$

Indeed, from (18) we deduce $K \leqslant K_i$. But $K_i \leqslant L$ follows from (19), and from this we obtain

$$K_i = \mathrm{Ker}\, I_{K_i}^H \leqslant \mathrm{Ker}\, I_L^H = K. \tag{20}$$

We have seen that the decomposition (18) corresponds to the decomposition

$$I_{L/K}^{H/K} = \bigoplus_{j=1}^{t} n_j \tau_j$$

with

$$T_j = \pi_j O \pi.$$

Hence $L \in$ Ek T_i implies

$$L \in \mathrm{Ek}\, \tau_i \qquad \mathrm{Ker}\, \tau_i = 1 \in H/K = G. \tag{21}$$

From this we conclude that a necessary condition for L to be an epikernel of the irreducible representation T_i of H is that H/K has faithful irreducible representations. Furthermore, L is an epikernel of the irreducible representation T_i of H if and only if L/K is an epikernel of the faithful irreducible representation τ_i of the finite group H/K. Thus everything is reduced to the study of faithful irreducible representations of finite groups.

5. Tables of equitranslation epikernels

If L is an equitranslation subgroup of a space group H (containing translations A), so is the core K of L. Thus $G = H/K$ is isomorphic to a crystallographic point group. The equitranslation epikernels such as L (of some irreducible representation T_i) of H are in a one-to-one correspondence with the epikernels J of the faithful irreducible representations of the finite group G. Thus ultimately we have to consider only the fourteen faithful irreducible representations of the twelve isomorphism classes of crystallographic point groups that have such representations. (The isomorphism classes S_4 and $\mathrm{S}_4 \times \mathrm{C}_2$ each have two of them.) This gives altogether forty pairs of groups $J < G$, and only twenty-five for which J is not only an epikernel but a maximal epikernel. The properties of the system near the transition depend on these pairs.

It may be seen from tables 1–18 that, in the cases where G is either S_4 or $\mathrm{S}_4 \times \mathrm{C}_2$, a pair of groups $L < H$ is not always sufficient to determine an irreducible representation of H. Thus, the transitions $\mathrm{O}_\mathrm{h}^1 \to \mathrm{C}_{2\mathrm{v}}^{14}$, $\mathrm{O}_\mathrm{h}^2 \to \mathrm{C}_{2\mathrm{v}}^{17}$, $\mathrm{O}_\mathrm{h}^3 \to \mathrm{C}_{2\mathrm{v}}^{16}$, $\mathrm{O}_\mathrm{h}^4 \to \mathrm{C}_{2\mathrm{v}}^{15}$, $\mathrm{O}_\mathrm{h}^{6,7} \to \mathrm{C}_{2\mathrm{v}}^{22}$, $\mathrm{O}_\mathrm{h}^8 \to \mathrm{C}_{2\mathrm{v}}^{21}$, $\mathrm{O}_\mathrm{h}^9 \to \mathrm{C}_{2\mathrm{v}}^{18}$ and $\mathrm{O}_\mathrm{h}^{10} \to \mathrm{C}_{2\mathrm{v}}^{19}$ determine the representations $\mathrm{T}_{1\mathrm{u}}$ and $\mathrm{T}_{2\mathrm{u}}$ of $\mathrm{S}_4 \times \mathrm{C}_2$ and these subgroups are maximal epikernels for both representations. In the following transitions, the subgroups are maximal epikernels for T_1 of S_4, but not for T_2 of S_4: $\mathrm{O}^{1-8} \to \mathrm{C}_2^3$, $\mathrm{T}_\mathrm{d}^{1,2,3} \to \mathrm{C}_\mathrm{s}^3$, $\mathrm{T}_\mathrm{d}^{2,4,6} \to \mathrm{C}_\mathrm{s}^3$, $\mathrm{O}_\mathrm{h}^{1,4,5,7,9} \to \mathrm{C}_{2\mathrm{h}}^3$ and $\mathrm{O}_\mathrm{h}^{2,3,5,7,10} \to \mathrm{C}_{2\mathrm{h}}^6$. Furthermore, $\mathrm{O}^{1-8} \to \mathrm{C}_1^1$, $\mathrm{T}_\mathrm{d}^{1-6} \to \mathrm{C}_1^1$ and $\mathrm{O}_\mathrm{h}^{1-10} \to \mathrm{C}_i^1$ determine T_1 and T_2 of S_4; $\mathrm{O}_\mathrm{h}^{1,3} \to \mathrm{C}_\mathrm{s}^1$, $\mathrm{O}_\mathrm{h}^{2,4} \to \mathrm{C}_\mathrm{s}^2$, $\mathrm{O}_\mathrm{h}^{5,6,9} \to \mathrm{C}_\mathrm{s}^3$, $\mathrm{O}_\mathrm{h}^{7,8,10} \to \mathrm{C}_\mathrm{s}^4$ and $\mathrm{O}_\mathrm{h}^{1-10} \to \mathrm{C}_1^1$ determine $\mathrm{T}_{1\mathrm{u}}$ and $\mathrm{T}_{2\mathrm{u}}$ of $\mathrm{S}_4 \times \mathrm{C}_2$, non of these subgroups being a maximal epikernel.

There are eighteen tables, one for each isomorphism class of crystallographic point group. In *column 1* the finite groups G are given and in *column 2* their faithful irreducible representations. *Column 3* shows their epikernels J and the number of conjugates for each of these given in *column 4*. Then, after a vertical line, we find the epikernels L (which are space groups) corresponding to those in the same row of column 3. In the first column of this second part the (geometric) crystal class of the space group is indicated. The Schoenflies superscripts of the space groups follow horizontally. Thus we find here in each column a space group (in the first row) and the equitranslation epikernels of the irreducible representations (below). The space groups of the first row are grouped using braces (⏞) into arithmetic crystal classes. If there are several isomorphic crystal classes, they (and the corresponding space groups) are separated by vertical lines. A horizontal broken line divides (for a given irreducible representation) the maximal epikernels (above) from the others.

We have adopted the Schoenflies symbols for their only advantage: shortness. More extended tables using the international notation are being prepared for a monograph. There it will be possible to indicate for each subgroup the orientation with respect to the axes of the supergroup H.

Table 1. C_1

C_1	A	C_1	1	C_1	1

Table 2. C_2

C_1	A	C_1	1	C_2	1	2	3	C_i	1	C_s	1	2	3	4
C_2	B	C_1	1	C_1	1	1	1	C_1	1	C_1	1	1	1	1

Table 3. C_3

C_1	A	C_1	1	C_3	1	2	3	4
C_3	E	C_1	1	C_1	1	1	1	1

Table 4. C_4

C_1	A	C_1	1	C_4	1	2	3	4	5	6	S_4	1	2
C_2	B	C_1	1	C_2	1	2	1	2	3	3	C_2	1	3
C_4	E	C_1	1	C_1	1	1	1	1	1	1	C_1	1	1

Table 5. $D_2 = C_2 \times C_2$

C_1	A	C_1	1	D_2	1	2	3	4	5	6	7	8	9	C_{2h}	1	2	4	5	3	6
C_2	B	C_1	1	C_2	1	1	2	2	2	3	3	3	3	C_i	1	1	1	1	1	1
C_2	B	C_1	1	C_2	1	2	1	2	3	1	3	3	3	C_2	1	2	1	2	3	3
C_2	B	C_1	1	C_2	1	1	2	2	3	3	3	3	3	C_s	1	1	2	2	3	4

C_{2v}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
C_s	1	2	2	2	2	2	2	2	2	2	3	4	4	3	3	4	4	3	4	3	4	4
C_2	1	2	1	1	2	1	2	1	2	1	1	2	1	3	3	3	3	3	3	3	3	3
C_s	1	1	2	1	2	2	1	2	2	2	3	3	4	1	2	1	2	3	4	3	4	3

Table 6. $C_6 = C_3 \times C_2$

C_1	A	C_1	1	C_6	1	2	3	4	5	6	C_{3h}	1	C_{3i}	1	2
C_2	B	C_1	1	C_3	1	2	3	3	2	1	C_3	1	C_3	1	4
C_3	E	C_1	1	C_2	1	2	2	1	1	2	C_s	1	C_i	1	1
C_6	E_2	C_1	1	C_1	1	1	1	1	1	1	C_1	1	C_1	1	1

Table 7. $C_4 \times C_2$

C_1	A	C_1	1	C_{4h}	1	2	3	4	5	6
C_2	B	C_1	1	C_4	1	3	1	3	5	6
C_2	B	C_1	1	C_{2h}	1	1	4	4	3	6
C_2	B	C_1	1	S_4	1	1	1	1	2	2
C_4	E	C_1	1	C_i	1	1	1	1	1	1
C_4	E	C_1	1	C_s	1	1	2	2	3	4

Table 8. $D_2 \times C_2 = C_2 \times C_2 \times C_2$

C_1	A	C_1	1	D_{2h}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
C_2	B	C_1	1	D_2	1	1	1	1	2	2	2	2	3	3	3	3	3	3	4	4
C_2	B	C_1	1	C_{2h}	1	4	4	4	2	4	1	5	5	5	4	5	2	5	5	5
C_2	B	C_1	1	C_{2v}	1	10	4	6	2	6	6	5	2	9	4	7	7	9	5	2
C_2	B	C_1	1	C_{2h}	1	4	1	4	4	4	5	4	1	4	2	1	4	5	5	5
C_2	B	C_1	1	C_{2v}	1	10	3	8	1	10	7	3	8	3	5	10	1	5	5	7
C_2	B	C_1	1	C_{2h}	1	4	4	4	1	5	4	4	5	5	5	5	2	4	5	2
C_2	B	C_1	1	$C_{2'v}$	1	10	4	6	4	9	4	8	2	9	2	7	7	6	5	9

17	18	19	20	21	22	23	24	25	26	27	28
5	5	6	6	6	6	7	7	8	8	9	9
3	3	3	6	3	6	3	6	3	6	6	3
16	17	14	16	15	17	18	19	20	22	21	22
2	5	1	1	4	4	3	6	3	3	6	6
12	12	11	13	11	13	18	19	20	21	21	20
6	3	3	6	3	6	3	6	3	6	6	3
14	15	14	16	15	17	18	19	20	22	21	22

Table 9. $C_6 \times C_2 = C_3 \times C_2 \times C_2$

C_1	A	C_1	1	C_{6h}	1	2
C_2	B	C_1	1	C_6	1	6
C_2	B	C_1	1	C_{3i}	1	1
C_2	B	C_1	1	C_{3h}	1	1
C_3	E	C_1	1	C_{2h}	1	2
C_6	E_2	C_1	1	C_2	1	2
C_6	E_2	C_1	1	C_i	1	1
C_6	E_2	C_1	1	C_s	1	1

Table 10. D_3

C_1	A	C_1	1	D_3	1	3	5	2	4	6	7	C_{3v}	1	3	2	4	5	6
C_2	B	C_1	1	C_3	1	2	3	1	2	3	4	C_3	1	1	1	1	4	4
D_3	E	C_2	3	C_2	3	3	3	3	3	3	3	C_s	3	4	3	4	3	4
		C_1	1	C_1	1	1	1	1	1	1	1	C_1	1	1	1	1	1	1

Table 11. D_4

C_1	A	C_1	1	D_4	1	2	3	4	5	6	7	8	9	10
C_2	B	C_1	1	C_4	1	1	2	2	3	3	4	4	5	6
C_2	B	C_1	1	D_2	1	3	2	4	1	3	2	4	8	9
C_2	B	C_1	1	D_2	6	6	5	5	6	6	5	5	7	7
D_4	E	C_2	2	C_2	1	2	1	2	1	2	1	2	3	3
		C_2	2	C_2	3	3	3	3	3	3	3	3	3	3
		C_1	1	C_1	1	1	1	1	1	1	1	1	1	1

D_{2d}	1	2	3	4	5	6	7	8	9	10	11	12
S_4	1	1	1	1	1	1	1	1	2	2	2	2
D_2	1	1	3	3	6	6	6	6	7	7	8	9
C_{2v}	11	13	11	13	1	3	8	10	20	21	18	19
C_2	1	1	2	2	3	3	3	3	3	3	3	3
C_s	3	4	3	4	1	2	2	2	3	4	3	4
C_1	1	1	1	1	1	1	1	1	1	1	1	1

C_{4v}	1	2	3	4	5	6	7	8	9	10	11	12
C_4	1	1	3	3	1	1	3	3	5	5	6	6
C_{2v}	1	8	3	10	3	10	1	8	20	21	20	21
C_{2v}	11	11	11	11	13	13	13	13	18	18	19	19
C_s	1	2	2	2	2	2	1	2	3	4	3	4
C_s	3	3	3	3	4	4	4	4	3	3	4	4
C_1	1	1	1	1	1	1	1	1	1	1	1	1

Table 12. A_4

C_1	A	C_1	1	T	1	4	3	5	2
C_3	E	C_1	1	D_2	1	4	8	9	7
A_4	T	C_3	4	C_3	4	4	4	4	4
		C_2	3	C_2	1	2	3	3	3
		C_1	1	C_1	1	1	1	1	1

Table 13. S_4

C_1	A	C_1	1	O	1	2	6	7	3	4	5	8	T_d	1	4	2	5	3	6
C_2	B	C_1	1	T	1	1	4	4	2	2	3	5	T	1	1	2	2	3	5
D_3	E	C_2	3	D_4	1	5	8	4	9	10	9	10	D_{2d}	1	2	9	10	11	12
		C_1	1	D_2	1	1	4	4	7	7	8	9	D_2	1	1	7	7	8	9
S_4	T_1	C_4	3	C_4	1	3	4	2	5	6	5	6	S_4	1	1	2	2	2	2
		C_3	4	C_3	4	4	4	4	4	4	4	4	C_3	4	4	4	4	4	4
		C_2	6	C_2	3	3	3	3	3	3	3	3	C_s	3	4	3	4	3	4
		C_1	1	C_1	1	1	1	1	1	1	1	1	C_1	1	1	1	1	1	1
S_4	T_2	D_2	3	D_2	6	6	5	5	8	9	7	7	C_{2v}	11	13	20	21	18	19
		D_3	4	D_3	7	7	7	7	7	7	7	7	C_{3v}	5	6	5	6	5	6
		C_2	6	C_2	3	3	3	3	3	3	3	3	C_s	3	4	3	4	3	4
		C_1	1	C_1	1	1	1	1	1	1	1	1	C_1	1	1	1	1	1	1

Table 14. $D_6 = D_3 \times C_2$

C_1	A	C_1	1	D_6	1	2	3	4	5	6	D_{3d}	1	2	3	4	5	6
C_2	B	C_1	1	C_6	1	2	3	4	5	6	C_{3i}	1	1	1	1	2	2
C_2	B	C_1	1	D_3	1	3	5	5	3	1	D_3	1	1	2	2	7	7
C_2	B	C_1	1	D_3	2	4	6	6	4	2	C_{3v}	2	4	1	3	5	6
D_3	E	C_2	3	D_2	6	5	5	6	6	6	C_{2h}	3	6	3	6	3	6
		C_1	1	C_2	1	2	2	1	1	2	C_i	1	1	1	1	1	1
D_6	E_1	C_2	3	C_2	3	3	3	3	3	3	C_2	3	3	3	3	3	3
		C_2	3	C_2	3	3	3	3	3	3	C_2	3	4	3	4	3	4
		C_1	1	C_1	1	1	1	1	1	1	C_1	1	1	1	1	1	1

C_{6v}	1	2	3	4	D_{3h}	1	2	3	4
C_6	1	1	6	6	C_{3h}	1	1	1	1
C_{3v}	1	3	3	1	D_3	1	1	2	2
C_{3v}	2	4	2	4	C_{3v}	1	3	2	4
C_{2v}	11	13	12	12	C_{2v}	14	16	14	16
C_2	1	1	2	2	C_s	1	1	1	1
C_s	3	4	4	3	C_2	3	3	3	3
C_s	3	4	3	4	C_2	3	4	3	4
C_1	1	1	1	1	C_1	1	1	1	1

Table 15. $D_4 \times C_2$

C_1	A	C_1	1	D_{4h}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
C_2	B	C_1	1	D_4	1	1	1	1	2	2	2	2	5	5	5	5	6	6	6	6	9	9	10	10
C_2	B	C_1	1	C_{4h}	1	1	3	3	1	1	3	3	2	2	4	4	2	2	4	4	5	5	6	6
C_2	B	C_1	1	C_{4v}	1	5	2	6	2	6	1	5	7	3	8	4	8	4	7	3	9	10	11	12
C_2	B	C_1	1	D_{2h}	1	3	4	2	9	12	13	10	1	3	4	2	9	12	13	10	25	26	28	27
C_2	B	C_1	1	D_{2d}	1	2	1	2	3	4	3	4	2	1	2	1	4	3	4	3	11	11	12	12
C_2	B	C_1	1	D_{2h}	19	20	21	22	19	20	21	22	20	19	22	21	20	19	22	21	23	23	24	24
C_2	B	C_1	1	D_{2d}	5	6	7	8	7	8	5	6	5	6	7	8	7	8	5	6	9	10	9	10
D_4	E	C_2	2	C_{2h}	1	4	4	4	5	5	2	5	1	4	4	4	5	5	2	5	3	6	3	6
		C_2	2	C_{2h}	3	6	3	6	3	6	3	6	6	3	6	3	6	3	6	3	3	3	6	6
		C_1	1	C_i	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D_4	E	C_2	2	C_{2v}	1	4	6	10	2	7	7	9	1	4	6	10	2	7	7	9	20	22	22	21
		C_2	2	C_{2v}	14	16	15	17	14	16	15	17	16	14	17	15	16	14	17	15	18	18	19	19
		C_1	1	C_s	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	3	3	4	4

Table 16. $A_4 \times C_2$

C_1	A	C_1	1	T_h	1	2	6	3	4	5	7
C_2	B	C_1	1	T	1	1	4	2	2	3	5
C_3	E	C_1	1	D_{2h}	1	2	15	23	24	25	27
C_6	E_2	C_1	1	D_2	1	1	4	7	7	8	9
A_4	T	C_2	3	C_{2h}	1	4	5	3	6	3	6
		C_3	4	C_{3i}	2	2	2	2	2	2	2
		C_1	1	C_i	1	1	1	1	1	1	1
$A_4 \times C_2$	T_u	D_2	3	C_{2v}	1	10	5	18	19	20	21
		C_3	4	C_3	4	4	4	4	4	4	4
		C_2	3	C_s	1	2	2	3	4	3	4
		C_1	1	C_1	1	1	1	1	1	1	1

Table 17. $S_4 \times C_2$

C_1	A	C_1	1	O_h	1	2	3	4	5	6	7	8	9	10
C_2	B	C_1	1	O	1	1	2	2	3	3	4	4	5	8
C_2	B	C_1	1	T_h	1	2	1	2	3	3	4	4	5	7
C_2	B	C_1	1	T_d	1	4	4	1	2	5	2	5	3	6
D_3	E	C_2	3	D_{4h}	1	4	9	12	17	18	19	20	17	20
		C_1	1	D_{2h}	1	2	1	2	23	23	24	24	25	27
D_6	E_1	C_2	3	D_4	1	1	5	5	9	9	10	10	9	10
		C_2	3	D_{2d}	1	2	2	1	9	10	9	10	11	12
		C_1	1	D_2	1	1	1	1	7	7	7	7	8	9
S_4	T_1	C_4	3	C_{4h}	1	3	2	4	5	5	6	6	5	6
		C_3	4	C_{3i}	2	2	2	2	2	2	2	2	2	2
		C_2	6	C_{2h}	3	6	6	3	3	6	3	6	3	6
		C_1	1	C_i	1	1	1	1	1	1	1	1	1	1
S_4	T_2	D_2	3	D_{2h}	19	22	20	21	25	26	28	27	23	24
		D_3	4	D_{3d}	5	6	6	5	5	6	5	6	5	6
		C_2	6	C_{2h}	3	6	6	3	3	6	3	6	3	6
		C_1	1	C_i	1	1	1	1	1	1	1	1	1	1
$S_4 \times C_2$	T_{1u}	D_4	3	C_{4v}	1	6	7	4	9	10	11	12	9	12
		D_2	6	C_{2v}	14	17	16	15	20	22	22	21	18	19
		D_3	4	C_{3v}	5	6	6	5	5	6	5	6	5	6
		C_2	6	C_s	3	4	4	3	3	4	3	4	3	4
		C_2	3	C_s	1	2	1	2	3	3	4	4	3	4
		C_1	1	C_1	1	1	1	1	1	1	1	1	1	1
$S_4 \times C_2$	T_{2u}	D_4	3	D_{2d}	5	8	5	8	11	11	12	12	9	10
		D_2	6	C_{2v}	14	17	16	15	20	22	22	21	18	19
		D_3	4	D_3	7	7	7	7	7	7	7	7	7	7
		C_2	6	C_2	3	3	3	3	3	3	3	3	3	3
		C_2	3	C_s	1	2	1	2	3	3	4	4	3	4
		C_1	1	C_1	1	1	1	1	1	1	1	1	1	1

Table 18. $D_6 \times C_2 = D_3 \times C_2 \times C_2$

C_1	A	C_1	1	D_{6h}	1	2	3	4
C_2	B	C_1	1	D_6	1	1	6	6
C_2	B	C_1	1	C_{6h}	1	1	2	2
C_2	B	C_1	1	C_{6v}	1	2	3	4
C_2	B	C_1	1	D_{3d}	1	2	1	2
C_2	B	C_1	1	D_{3h}	1	2	2	1
C_2	B	C_1	1	D_{3d}	3	4	4	3
C_2	B	C_1	1	D_{3h}	3	4	3	4
D_3	E	C_2	3	D_{2h}	19	20	17	17
		C_1	1	C_{2h}	1	1	2	2
D_6	E_1	C_2	3	D_2	6	6	5	5
		C_2	3	C_{2v}	11	13	12	12
		C_1	1	C_2	1	1	2	2
D_6	E_1	C_2	3	C_{2h}	3	6	3	6
		C_2	3	C_{2h}	3	6	6	3
		C_1	1	C_i	1	1	1	1
D_6	E_1	C_2	3	C_{2v}	14	16	16	14
		C_2	3	C_{2v}	14	16	14	16
		C_1	1	C_s	1	1	1	1

6. Concluding remarks

Permutation representations reveal themselves as a powerful tool in the study of phase transitions (or transitions between two states of different symmetry). There remain a few unanswered questions about epikernels of finite index. However, the most interesting physical and mathematical problems ahead concern the generalization of the results so far obtained to epikernels of infinite index and the standard forms of the thermodynamic potentials.

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References

Ascher E 1966a *Helv. Phys. Acta* **39** 466–76
—— 1966b *Phys. Lett.* **20** 352–4
—— 1971 *Les Transitions de Phase, 13e Cours Association Vaudoise Chercheurs En Physique* pp 34–80
Ascher E and Kobayashi J 1977 *J. Phys. C: Solid St. Phys.* **10** 1349–63
Birman J L 1966 *Phys. Rev. Lett.* **17** 1216–9
Curie P 1894 *J. Physique* **5** 289

Janovec V, Dvorak V and Petzelt J 1975 *Czech. J. Phys.* B **25** 1362–96
MacLane S and Birkhoff G 1967 *Algebra* (London:Collier-MacMillan)
McDowell R S 1965 *J. Molec. Spectrosc.* **17** 365–7
Melvin M A 1956 *Rev. Mod. Phys.* **28** 18–44
Murray-Rust P, Burgi H B and Dunnitz J D 1975
Sonin A S and Zhëludev I S 1959 *Kristallogr.* **4** 487–97
Zhëludev I S and Shuvalov L A 1956 *Kristallogr.* **1** 681