



The Case of Piaget's Group INRC<sup>\*</sup>

---

Edgar Ascher

Département de Physique Théorique

Université de Genève

1211 GENEVE 4, Switzerland

UGVA-DPT 1983/03-385

Abstract

What is it that distinguishes Piaget's transformations  $N, R$  and  $C$  from the rest of the  $16!$  transformations of the  $16$  binary propositional operations? Here INRC is considered as a subgroup of the group  $\mathcal{M}_2$  of all automorphisms and dual automorphisms of the free Boolean algebra with two generators. This group is isomorphic to  $S_4 \times C_2$ . Its elements are given explicitly. Many other psychologically relevant subgroups of  $\mathcal{M}_2$  play an important role. They are discussed and their connections shown. Particular attention is given to involutions, even if the view that they constitute the sole representation of reversibility is abandoned. Piaget's transformation  $R$  turns out not to be the inverse operation of relations. The group of automorphisms, dual automorphisms, anti-automorphisms of the algebra of binary relations on a finite set is found. A crystallographic presentation of these groups is given and related work by Bart, Leresche and Pólya is discussed.

---

<sup>\*</sup> Supported by the Swiss National Science Foundation

Algebraic Models

My purpose in this paper is to dispel some misconceptions about Piaget's group INRC and thereby to open the way for a fruitful use of groups of so-called logical transformations and their subsequent transfer to other domains of cognitive activities. The group INRC appears for the first time in Piaget (1950). Many comments and discussions have appeared since ; but much of it is subtly vague, at best. If one wants to be more precise, it behooves to start with making the mathematical environment of that group more precise. This is not to say that the psychological and epistemological problems Piaget wanted to tackle are in fact mathematical problems. But it is useless, and my indeed be dangerous, to ask of psychology what mathematics by itself can accomplish. On the other hand, it is clear that one cannot reduce psychology to mathematics. The psychologist has to select the adequate mathematical structure for his problem. (This, of course, includes the possibility of deciding not to attempt any mathematical formalisation). But then he must be sufficiently aware of the mathematical possibilities which can be related to his subject matter. Sparse mathematics may be blind to interesting aspects of behaviour, adequate mathematics is full of suggestions for experiment. And that is another reason to discuss some of the various mathematical questions

raised by the investigation of the role played by the group INRC in the genetic psychology of logical operation "Our main goal is [to contribute to] the construction of an adequate mathematical model, wherein one can make deductions and inferences strictly from the mathematical structure itself, and then emerge from the mathematical structure with interpretation hypothesis .." (Hoffman, 1980, p. 422).

The mathematical structures we want to discuss belong to algebra (and to its most elementary parts at that). Again this does not mean that other mathematical structures may not be useful for formalizing problems that arrive in genetic psychology. But Piaget has favoured, what he calls, "logical formalization" : "... logical formalization is absolutely essential every time we can carry out some formalization ; ..." Piaget 1970, p. 10).

There are, however, at least two styles of logical formalization : the algebraic one and that of the theory of deduction. As far as Piaget is concerned, it seems correct to say that "... the author, joining the tradition of algebraic logic of Boole, Peirce and Schröder, takes an algebraic view of the subject rather than the one of the theory of deduction" (Beth 1950, p. 259). Within the context of Beth's virulent attack against the 'Traité de Logique' (Piaget, 1949), this evocation, of what to Beth then appeared to be the rear-guard of a defeated tradition, was clearly meant to depreciate Piaget's book. Nevertheless, it remains true that "logical formalization"

for Piaget does mean, in the first place, algebraic formalization. Indeed, he says (Piaget, 1967, p. 269). "In view of the present lack of numerical units in psychology, the usefulness of logical models consists in providing a possible image of the very operations of the subject, as well as of the operational structures ('groupings', groups, lattices, etc) which these operations form between themselves".

What he has in mind then, are qualitative models and not quantitative ones ; not the mathematical disciplines related to psychometrics, but algebra.

In this paper, we shall deal with some algebraic structures of logical operations. Such structures may be thought to represent the "algebra of competence" which could be detected in the pattern of actions of a person. Thus, actions in the most diverse fields may be related as are the elements of Klein's four-group  $D_2$ , of which Piaget's INRC is a realization as group of logical transformations. Here, however, we shall not study these other occurrences, but rather explore other groups of logical transformations. (Appendix 1 is an exception ; there, a crystallographic representation of these groups is presented). The preconception that INRC is the only group of logical transformations (in addition, of course, to its subgroups) may easily conceal other more complicated regularities in all sorts of situation. One among the most basic experiences for instance, the non-commutativity of operations, cannot be captured by a commutative group like  $D_2$ .

### The Group INRC and the Automorphisms of $FB(2)$

Piaget, (1972, p. XIV), "became aware of the existence of this group [INRC] within the propositional operations before 1949 (believing, moreover, that they were known), that is at a time when logicians still had not taken any interest in it."

This group, INRC, operates on the sixteen binary propositional operations (BPO). But there are  $16!$  that is roughly  $2.10^{13}$  transformations of the sixteen BPO and not only four. At first, the whole symmetric group  $S_{16}$ , the group of all permutations of sixteen items, has to be taken into account.

However, most of these transformations are not interesting. It seems natural to consider the sixteen BPO not simply as any sixteen items, but to take into account how these BPO are related with each other. If one looks at the manner in which the sixteen BPO can be obtained from two of them by using conjunctions, disjunctions and negations, then one is led to consider the BPO as elements of a Boolean algebra, and more precisely, of a free Boolean algebra  $FB(2)$  with two generators. Consequently, one is then led to take into account only those transformations that "respect" this structure, in a sense which will be made clear presently.

Bart (1971) has imposed quite different conditions on the transformations of the elements of  $FB(2)$  ; the Boolean structure is not respected (cf. Appendix 3).

Let us recall here what a Boolean algebra is and what is meant by free. We do this in a way which permits easy comparison with operations on the set of relations we shall define below. Furthermore, making the algebraic ingredients of Boolean algebras explicit, may suggest experiments concerning these features and about the gradual building-up of Boolean algebras from them.

A Boolean algebra

$$\mathcal{B} = \langle B, \wedge, \vee, ', 0, 1 \rangle$$

is a set  $B$  with two binary operations  $\wedge$  (meet) and  $\vee$  (join) one unary operation  $'$  (complement) and two nullary operations (i.e. special elements)  $0$  and  $1$ . These operations have properties stated and explained below.

(i) First consider  $\mathcal{B}_\wedge = \langle B, \wedge, 1, 0 \rangle$ .

( $\alpha$ ) The operation  $\wedge$  is associative, i.e. for all elements  $x, y, z$  of  $B$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z. \quad (1)$$

( $\beta$ ) It has a unit element  $1$ ; this means

$$x \wedge 1 = 1 = 1 \wedge x \quad (2)$$

for all  $x$ .

Properties ( $\alpha$ ) and ( $\beta$ ) are usually summarized by saying that  $\mathcal{B}_\wedge$  is a monoid. However, this monoid has further properties.

( $\gamma$ ) It has a null element  $0$  :

$$x \wedge 0 = 0 = 0 \wedge x. \quad (3)$$

( $\delta$ ) It is commutative :

$$x \wedge y = y \wedge x. \quad (4)$$

( $\epsilon$ ) It is idempotent :

$$x \wedge x = x \quad (5)$$

for any element  $x$  of  $B$ .

(ii)  $\mathcal{B}_\vee = \langle B, \vee, 0, 1 \rangle$

is a monoid (unit element  $0$ ) with null element  $1$ , that is furthermore commutative and idempotent.

(iii) The operation  $'$  is characterized by

$$(\alpha) \quad (x')' = x, \quad (6)$$

$$(\beta) \quad x \wedge x' = 0, \quad x \vee x' = 1 \quad (7)$$

(iv) The operations  $\wedge$  and  $\vee$  are linked in many ways.

( $\alpha$ ) Absorption :

$$x \wedge (x \vee y) = x = x \vee (x \wedge y), \quad (8)$$

( $\beta$ ) left distributivity : (Right distributivity then follows from commutativity).

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad (9)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad (10)$$

( $\gamma$ ) De Morgan equalities

$$(x \wedge y)' = x' \vee y', \quad (11)$$

$$(x \vee y)' = x' \wedge y', \quad (12)$$

$$1' = 0, \quad 0' = 1. \quad (13)$$

We may also say that, (11) (12) (13) and (6) express the fact that  $'$  is an isomorphism of the monoids  $\mathcal{B}_\wedge$  and  $\mathcal{B}_\vee$ .

A Boolean algebra is free with  $n$  generators if it contains exactly all different elements that can be formed from the  $n$  generators by performing the operations  $\wedge$ ,  $\vee$  and  $'$ .

If we then consider the sixteen BPO as elements of the free Boolean algebra  $FB(2)$  with two elements, it is reasonable to take into account not simply transformations but only transformations that conserve the Boolean structure, that is Boolean automorphisms. Such an automorphism  $\phi$  is a one-to-one mapping of the set  $B$  onto itself such that

$$\phi(x \wedge y) = \phi x \wedge \phi y, \quad (14)$$

$$\phi(x \vee y) = \phi x \vee \phi y, \quad (15)$$

$$\phi(x') = (\phi x)', \quad (16)$$

$$\phi 0 = 0, \quad \phi 1 = 1. \quad (17)$$

This limitation to automorphisms turns out to be altogether too drastic, since the transformations  $N$  and  $C$ , as defined and used by Piaget, are not automorphisms; they do not, for instance, conserve the special elements  $0$  and  $1$ , as would be required by (17). But it will be easy to get hold of these and similar elements once the automorphisms have been found.

It is not difficult to find the automorphisms of a Boolean algebra. One has to take into account the fact that every finite Boolean algebra is isomorphic to the Boolean algebra of all subsets of a finite set; the operations are the union and the intersection of sets, the set theoretic complement, the void set and the original set. If the set has  $n$  elements, the corresponding Boolean algebra will be noted  $B(n)$ . It has  $2^n$  elements. A free Boolean algebra with  $n$  generators,  $FB(n)$ ,

has  $2^p$  (with  $p = 2^n$ ) elements and is isomorphic to  $B(2^n)$ . In the case of the sixteen BPO we have  $n = 2 : FB(2) \approx B(4)$ .

The set of all automorphisms is a group. (A group is a monoid in which all elements have an inverse.) The group of all automorphisms of a finite Boolean algebra  $B(n)$  can be obtained from, and is isomorphic to, the symmetric group  $S_n$ , the group of all permutations of the elements of the corresponding set. More precisely, it is a subgroup of the group  $S_p$  ( $p = 2^n$ ) of all permutations of the  $2^n$  elements of  $B(n)$ , that is isomorphic to  $S_n$ . In particular,  $FB(n)$  the free Boolean algebra with  $n$  generators has  $2^p$  ( $p = 2^n$ ) elements and its automorphisms group  $\mathcal{A}_n$  is a subgroup of  $S_q$  ( $q = 2^p$ ,  $p = 2^n$ ) that is isomorphic to  $S_p$ .

In the case of the free Boolean algebra  $FB(2)$ , the automorphism group  $\mathcal{A}_2$  is isomorphic to  $S_4$  and has twenty-four elements. For the subsequent discussions, we need to know how these transformations operate on the sixteen elements of  $FB(2)$ . It is also worth while to understand how to obtain them. We therefore introduce a short notation for each of the elements of  $FB(2)$ . We derive it from the notation for the binary propositional operations introduced by Łukasiewicz. He writes, for instance,  $Apq$  for  $p \vee q$  and  $Kpq$  for  $p \wedge q$ . We note the corresponding elements of  $FB(2)$  by  $a$  and  $k$ , respectively and thus obtain the notation contained in Table 1. Furthermore, to each element we associate (column 3) a subset of a set of four elements, whose elements are designated by the same letters as the four elements  $k, l, m$  and  $x$ . This establishes an isomorphism between  $FB(2)$  and the subsets of the set  $\{k, l, m, x\}$  in

which there corresponds to each subset the element of  $BL(2)$  that is the disjunction of the elements of the subset. Thus, for instance,  $\{k,l,m\}$  corresponds to  $k \vee l \vee m = a$ .

---

Insert Table 1 about here

---

Let us now see how, from the group  $S_4$  of the permutations of the four elements  $k, l, m$  and  $x$ , one obtains an isomorphic subgroup of  $S_{16}$ , the group of all permutations of the sixteen elements of  $FB(2)$ . Let us take, for instance, the transformation  $(kx)(lm)$  that exchanges first  $l$  and  $m$ , and then  $k$  and  $x$ . This transformation induces, for example, a transformation of the set  $\{k,l\}$  into the set  $\{m,x\}$ , that is (according to Table 1) of the element  $f$  into the element  $e$ ; the set  $\{k,l,x\}$  is transformed into  $\{k,m,x\}$ , that is  $b$  is transformed into  $c$  - and soon. We thus find that  $(kx)(lm) \in S_4$  induces

$$(kx)(lm)(ad)(bc)(if)(hg)(e)(j)(v)(o) \in S_{16}.$$

It may be seen that this is Piaget's transformation  $R$  - e.g. in Piaget (1972, p. 254).

This way, we obtain the whole group  $\mathcal{A}_2$  of automorphisms of  $FB(2)$  shown in Table 2. The elements of  $\mathcal{A}_2$  are represented as products of cycles. The cycles are applied to the elements of  $FB(2)$  from right to left. A cycle  $(lkxm)$ , for instance, transforms  $l$  into  $k$ ,  $k$  into  $x$ ,  $x$  into  $m$ , and  $m$  into  $l$  :  
 $(lkxm) = (lk)(kx)(xm)$

---

Insert Table 2 about here

---

Note that  $\mathcal{A}_2$  contains elements of orders two, three and four. The 24 elements are divided into five groups ; these are conjugate classes of  $S_4$  defined and used below. We have omitted to indicate how the automorphisms transform the elements  $v$  and  $o$ . According to (17) they remain unchanged. Thus,  $(v)(o)$  should be added to each automorphism.

Some of these automorphisms have been given special symbols  $(P,Q,R,U,T,V)$ . They are involutions. (there are three other involutions in  $\mathcal{A}_2$ ). Involutions are transformations  $X$  of order two :  $X^2 = I$ . (Here,  $I$  is the identity transformation). Such transformations are equal to their inverse :  $X = X^{-1}$ . Piaget calls them also 'reciprocities'.

The important role played by involutions in the formalization of Piaget's psychology and epistemology has been stressed by several authors (see e.g. Wermus (1972)). But the groups of transformations considered were always groups consisting of involutions (i.e. Boolean groups). The group  $\mathcal{A}_2$  contains 15 such groups and the  $\mathcal{M}_2$  relevant to the discussion of the transformations of the BPO (cf. subsequent section) contains 48 of them, of which 25 are isomorphic to INRC. Thus, INRC has no special status a priori as a group of logical transformations.

Involutions are supposed to represent the reversibility of operations. However, "this 'strong' view of reversibility is derived not from extensive empirical examinations of reversibility behaviors but rather from a consideration of the reversibility indicated in the INRC group" (Bart, 1971, p. 542). As soon

as the idea of the uniqueness of INRC as group of logical transformations is abandoned, other interpretations may arise, which, nevertheless, are "compatible with the nature of such transformations as I and N" as Bart (ibid.) rightly insists.

It is clear that transformations of orders higher than two play a role in the functioning of intelligence. In such cases, reversibility simply corresponds to the existence of an inverse transformation, that however need not be equal to the original one. By the way, there are cases in literature, where actions have been modeled as involutions, although repeating that action was by no means equivalent to undoing it.

Nevertheless, giving up the strong view of reversibility, the exclusive privilege of involutions to represent reversibility, does not mean abandoning involutions altogether. They still play an important role in the groups underlying a person's intellectual (and other) activities. The group  $\mathcal{A}_2$  of automorphisms of  $LB(2)$  is a good illustration of what may happen. This group is not a group of involutions, but a group generated by involutions, the three involutions U, T and V for instance. That is, every element of  $\mathcal{A}_2$  can be written as a product of these three elements (generators), as shown in the column 'generators' of Table 1. It can be seen there that no element is the product of more than four factors.

A more complicated structure than that of Boolean groups arises because the generating involutions do not commute. Non-

commutativity and the appearance of higher order (i.e. non-involutive) operations are probably closely related genetically. Now, non-commutativity is certainly a basic experience in the growth of intelligence. Below, in the discussion of subgroups, we shall find a typical example of non-commutativity.

Involutive generators of  $\mathcal{A}_2$  other than U, T and V may be selected. Some of these selections will be discussed below. Depending on the choice of generators, some automorphisms appear more complicated than others, since they are represented as products of many factors. Such a notational complication may reveal psychological complexity. One may even require that it be so; that is, that the generators be selected in such a way that the genetically simple appears as simply generated. There is, of course, also the possibility of choosing as generators elements other than involutions. The possibilities are numerous. To point this out is all we can do here. The algebraic model must be adopted to actual experience.

We conclude this section by remarking that quite generally,  $\mathcal{A}_n$  the group of automorphisms of the free Boolean algebra  $BL(n)$  with  $n$  generators, is isomorphic to  $S_p$  ( $p = 2n$ ) and that it is generated by  $2^n - 1$  involutions. It is also useful to know, that it may be viewed as a group of crystallographic transformations of a  $(2^n - 1)$ -dimensional crystal. In the case we are mainly interested in ( $n = 2$ ), the crystal is an ordinary 3-dimensional cubic crystal. More about the crystallographic representation may be found in Appendix 1.

Dual Automorphisms

Piaget's transformations N and C are not automorphisms of FB(2), but dual automorphisms (d-automorphisms for short), that is, transformations that fulfil the following conditions :

$$\Psi(x\wedge y) = \Psi x \vee \Psi y, \tag{18}$$

$$\Psi(x\vee y) = \Psi x \wedge \Psi y, \tag{19}$$

$$\Psi(x') = (\Psi x) ' \tag{20}$$

$$\Psi 0 = v, \Psi v = 0. \tag{21}$$

We may now say that equations (11), (12) and (13) express the fact that ' is a d-automorphism. Furthermore, it is time to note that Piaget's transformations N, the Boolean negation, is the operation ' :

$$Nx = x' ; \tag{22}$$

thus, it is indeed a d-automorphism. It has the further property of commuting with all automorphisms ; equations (16) and (22) give together

$$\phi N(x) = N\phi(x) \tag{23}$$

It is important to note that the product of an automorphism and a d-automorphism is a d-automorphism. Piaget's C, which is equal to RN, is an example. Furthermore, as it is easy to convince oneself, the product of two d-automorphisms is an automorphism. (The d-automorphisms do not form a group). It follows that in a group containing automorphisms and d-automorphisms, there is the same number of both of them ; the order

of such a group is necessarily even. We shall call such a group, a group of "mixed automorphisms". More precisely, such a group is the extension of its normal subgroup of automorphisms by the group  $C_2$  of order two. Particular cases of extensions of one group by another are their semi-direct products and the direct product.

If among the d-automorphisms of a group of mixed automorphisms, there is one involution, say X, that commutes with all automorphisms, then the group is a direct product of its subgroup of automorphisms with the group of order two generated by that element. Equation (12) shows N is such an element. Therefore, the group  $\mathcal{M}_2$  of all mixed automorphisms of BL(2) is the direct product of  $\mathcal{A}_2$  and the group  $\langle N \rangle$  generated by N ; it is isomorphic to  $S_4 \times C_2$  :

$$\mathcal{M}_2 = \mathcal{A}_2 \times \langle N \rangle \simeq S_4 \times C_2 \tag{24}$$

In other words, each d-automorphism is the product of an automorphism with N. (But in a subgroup of  $\mathcal{M}_2$  these automorphisms may be absent). The action of N is known from (22) ; thus, all d-automorphisms can be computed by multiplication of permutations. The result is shown in Table 3.

---

Insert Table 3 about here

---

Since  $\mathcal{A}_2$  was generated by three involutions, it follows from the construction, that  $\mathcal{M}_2$  is generated by four of them, U, T, V and N for example. However, we shall see that it can be

generated by three involutions only, and in a given context this will appear as a natural way of generation.

The 24  $d$ -automorphisms form five conjugate classes of  $S_4 \times C_2$ . Note that there is a class of elements of order six. Again, we have not indicated the action of the  $d$ -automorphisms on  $v$  and on  $o$ . According to (21),  $v$  and  $o$  are exchanged in all cases. Thus,  $(vo)$  should be added to each  $d$ -automorphism.

Generally, the group  $\mathcal{M}_n$  of all mixed automorphisms of  $BL(n)$  is given by

$$\mathcal{M}_n = \mathcal{A}_n \times \langle N \rangle \simeq S_p \times C_2 \quad (25)$$

(with  $p = 2^n$ ). It can be generated by  $2^n - 1$  involutions.

We have thus found a group  $\mathcal{M}_2$  of 48 elements operating on  $BL(2)$ . For various reasons it may be tempting to want to deal with smaller and simpler groups. However, reasons for doing so have to be advanced, at least tentatively. The restriction must not simply be obtained through unawareness of other possibilities or by decree. In the following two sections, we seek for criteria permitting to restrict the group  $\mathcal{M}_2$  to some of its subgroups.

Restricted Mobility, Subgroups

To suppose the whole group  $\mathcal{M}_2$  of mixed automorphisms available to somebody, is to assume a mobility of thought, that probably very few adults ever attain.

The sixteen binary propositional operations constitute the free Boolean algebra with two generators  $FB(2)$ . There are, however, several possibilities of choosing these generators : essentially three possibilities, since with each generator its complement also occurs. We have thus the three systems of generators :

$$\begin{aligned} \{p,q\} & \quad \text{or} \quad \{i,h\} \\ \{p, p \leftrightarrow q\} & \quad \text{or} \quad \{i,e\} \\ \{p \leftrightarrow q, q\} & \quad \text{or} \quad \{e,h\} \end{aligned}$$

The group  $\mathcal{A}_2$  of all automorphisms (and, a fortiori, the group  $\mathcal{M}_2$ ) exchanges the systems of generators. The automorphisms,  $U$  and  $V$ , for instance, transform the pair  $\{i,h\}$  into  $\{e,h\}$  and  $\{i,e\}$  respectively. It seems reasonable, therefore, to investigate those transformations that do not change one chosen system of generators into another. As for the choice of the system, it appears most natural to take  $\{p,q\}$  (the "standard generator system"). We are thus selecting now those transformations contained in Tables 2 and 3 that permute the elements  $i,h,f$  and  $g$  only among themselves (and, by the way, do the same to the remaining 12 elements of  $FB(2)$ ). We find the 16 transformations detailed in the second and third column of Table 4.

---

Insert Table 4 about here

---

The 16 elements form a subgroup  $G_1$  of  $\mathcal{M}_2$ ; they make up the invariance group of the generator system  $\{p,q\}$ . However, they act on the generators  $p$  and  $q$  only in eight distinguishable ways, as shown in the first column of Table 4. For each automorphism, there is a  $d$ -automorphism, that acts in the same way on the set  $\{p,q,p'q'\}$ , but differs, of course, in its action on the remaining elements of  $FB(2)$ . The  $d$ -automorphism  $C = \bar{R} = RN$ , for instance, reduces to the identity transformation when restricted to the above set. Thus, any automorphism and the  $d$ -automorphism resulting from it by multiplication with  $\bar{R}$  have the same effect on the generator system  $\{p,q\}$ . The action of  $G_1$  is the same as that of its subgroup of automorphisms  $G_1'$  (second column of Table 4) or, indeed, to that of the three other subgroups of  $G_1$  that are isomorphic to it; all are isomorphic to  $D_4$ . We shall refer to  $G_1$  and  $G_1'$  as "restricted groups" of respectively mixed automorphisms and automorphisms of  $FB(2)$ .

Recall that the dihedral group  $D_n$  ( $n \geq 3$ ) is a group of  $2n$  elements that is isomorphic to the group of symmetries of a regular polygone of  $n$  sides. It is not commutative and is generated by two involutions,  $x$  and  $y$ , obeying the defining equations

$$x^2 = y^2 = (xy)^n = I \tag{26}$$

For  $n = 2$  the equations also define a group, the commutative group  $D_2$ . Piaget's INRC is isomorphic to  $D_2$ . Groups isomor-

phic to  $D_2$  can be found in many situations and  $D_2$  has been given various names, of which "Klein's four-group", or simply "four-group" are found most frequently.

The fact that  $x$  and  $y$  do not commute for  $n \neq 2$  is shown most clearly by

$$xy = (yx)^{n-1} \tag{27}$$

an immediate consequence of (26). In columns three and four of Table 4, we have displayed  $Q$  and  $T$  (and also  $\bar{P}$  and  $T$ ) as generators. Eq. (27) specializes to

$$TQ = (QT)^3 \tag{28}$$

and this shows how the non-commutativity comes about. The operations of exchanging the two generators,  $p$  and  $q$ , (by  $T$ ) and that of exchanging  $p$  with its complement  $p'$  (by  $Q$ ) (i.e. negating one of the generators) do not commute. The automorphisms  $P$  that negates (complements) the other generator  $q$ , does not commute with  $T$  either. This may exemplify one of the most primitive experiences of non-commutativity: exchanging the two members of an ordered pair (this is an involution) and doing something to one member are operations that do not commute, and performing one after the other results in an operation that is not an involution, even though one starts with two involutions. It may be interesting to investigate early experiences of non-commutativity. The four subgroups of  $G_1$  isomorphic to  $D_4$  mentioned above are the groups generated by (for instance)  $Q$  and  $TN$ ,  $Q$  and  $T$ ,  $\bar{P}$  and  $TN$  and finally  $\bar{P}$  and  $T$ . Let us use the last group to illustrate the fact already mentioned, namely,

that a group of mixed automorphisms is not necessarily the direct product of its subgroup of automorphisms with the group  $\langle N \rangle$  generated by the Boolean negation  $N$ . In our group, there are four automorphisms  $I, \overline{TP}, T\overline{P}, T$  and  $\overline{PTP}$ ; the d-automorphisms  $\overline{P} = PN, T\overline{PT}, PT$  and  $\overline{TP}$  are not obtained by multiplying by  $N$  the automorphisms of the group (i.e. the transformations  $P, TPT, PT$  and  $TP$  do not belong to the group in question).

The whole group  $G_1$  is generated by adding to the involution generating  $D_4$  one more involution:  $N$  or  $\overline{R} = C$ ; both these d-automorphisms commute with all other elements. The structure of  $G_1$  is thus that of a direct product

$$G_1 \simeq D_4 \times C_2 \quad (29)$$

It must be emphasized that this is the group one should take into account. It is true that by concentrating on the operations acting only on the generators  $p$  and  $q$  and their complements we would obtain a group  $D_4$ . But then, we would miss the important relation between the elements of this group and the elements of the Boolean algebra  $FB(2)$ , namely, that they are automorphisms or d-automorphisms. As soon as we have shown this fact, there is no reason to select eight elements of  $D_4$  out of the 16 that constitute  $G_1 \simeq D_4 \times C_2$ .

It is plausible to think that the restricted groups  $G_1$  and  $G_1'$  represent the mobility of thought that a person of the formal stage (i.e. a non-specialized person) may attain. However, in the gradual building-up of competence, sub-groups certainly play an important role. We now proceed to discuss some of them.

We have already met with the group

$$G_7 = \{I, \overline{R}\} = \langle \overline{R} \rangle \quad (30)$$

generated by  $\overline{R}$ . It is the subgroup that leaves  $p$  and  $q$  as well as their complements unchanged. In a sense, this is the most simple group of mixed automorphisms one can imagine. From a psychological point of view, however, the status of such transformations is more problematic. On the face of it they are transformations that transform nothing; they act on unobserved, though observable features. Probably, such transformations are thematised relatively late.

Studying Table 4, we can characterize some other subgroups of  $G_1$ . The group that fixes the generator  $p$  (as well as  $p'$ ) and the one that fixes  $q$  and  $q'$  are respectively

$$G_p = \{I, P; \overline{R}, \overline{Q}\} = \langle P \rangle \times \langle \overline{R} \rangle \quad (31)$$

$$G_q = \{I, Q; \overline{R}, \overline{P}\} = \langle Q \rangle \times \langle \overline{R} \rangle \quad (32)$$

The group  $G_p$  is generated by  $P$  and  $\overline{R}$ ,  $G_q$  by  $Q$  and  $\overline{R}$ . The elements listed after the semicolon are d-automorphisms. The generators are exchanged (and their complements also) by

$$G_5 = \{I, T; \overline{R}, \overline{TR}\} = \langle T \rangle \times \langle \overline{R} \rangle \quad (33)$$

The two generators are changed into their complements by

$$G_4 = \{I, R; \overline{R}, N\} = \langle R \rangle \times \langle \overline{R} \rangle \quad (34)$$

This is Piaget's INRC. All these groups are isomorphic to Klein's four-group)  $D_2$ .

Generators are fixed or exchanged with their complements by

$$G_2 = G_P \vee G_Q = \{I, P, Q, R, N, \bar{P}, \bar{Q}, \bar{R}\} = \langle P \rangle \times \langle Q \rangle \times \langle N \rangle \quad (35)$$

Note that the meet of two groups is not their set-theoretic union (this would not be a group), but the group generated by the groups in question.

The join of two groups, however, is their set-theoretic intersection, which is a group. Finally, the group that exchanges generators and also their complements or changes generators into complements is

$$G_3 = G_4 \vee C_5 = \{I, R, T, TR; N, R, TN, TR\} = \langle R \rangle \times \langle T \rangle \times \langle N \rangle. \quad (36)$$

We shall meet this group when discussing structure preserving transformations of relations. Both the groups  $G_2$  and  $G_3$  are isomorphic to  $D_2 \times C_2 = C_2 \times C_2 \times C_2$ .

---

Insert Figure 1 about here

---

How these groups are related, is shown in the lattice of Figure 1 of subgroups of the group of mixed automorphisms  $\mathcal{M}_2$ . It is a sublattice of the lattice of all subgroups of  $\mathcal{M}_2$ . The groups on the right are groups of mixed automorphisms, those on the left are automorphisms. Such a presentation shows the position of INRC (boxed) among the other groups and it appears that this position is not exceptional. The lines linking a group to a supergroup (above) and a subgroup (below) may also

suggest pathways for the gradual building-up of these transformation groups during the development of intelligence.

All the subgroups of  $\mathcal{M}_2$  and of  $\mathcal{A}_2$  discussed so far concern the generator system  $\{i, h\}$ . The corresponding subgroups for the systems  $\{i, e\}$  and  $\{e, h\}$  are conjugate to those for  $\{i, h\}$ . Recall that the conjugate of a subgroup  $G$  of  $\mathcal{M}_2$  is a group consisting of all elements  $xgx^{-1}$  with  $x \in \mathcal{M}_2$  and  $g \in G$ . The set  $C_g = \{xgx^{-1} : x \in G\}$  of all elements conjugate to a given element  $g$ , is termed the conjugate class of  $g$ . Conjugate classes are either disjoint or coincide. Tables 3, 4 and 5 display this partition into conjugate classes..

If  $x$  is not in  $G$ , then the group  $xGx^{-1}$  may be different from  $G$  or equal to  $G$ . It is equal when  $G$  is a normal (self-conjugate) subgroup, otherwise it is different. In the lattice of Figure 1, the normal subgroups are placed in a circle. These groups are common to the three generator systems. The only case not obvious from the beginning is that of the group  $G_2 = \langle P \rangle \times \langle Q \rangle \times \langle N \rangle$ , and of its subgroups of automorphisms  $G_2'$ . These groups fix the generators or exchange them with their conjugates in any generator system.

A word of caution is needed for the groups  $G_5$  and their subgroup of automorphisms  $G_5'$  that exchange the generators and also their complements. They have six conjugates and not only three like the other non-normal subgroups. One should add to the subgroups already considered the subgroup  $G_6$  that exchanges a generator with the complement of the other :

$$G_6 = \{I, TP; C, TN\} = \langle TR \rangle \times \langle C \rangle.$$

The pair  $(G_5, G_7)$  is transformed by conjugation into the pairs, corresponding to the other generator systems.

With the help of the crystallographic translation of Appendix 1, all these groups may be visualized as transformations of a cube. The change of generator system, particularly, becomes easy to grasp.

In addition to the gradual building-up of the restricted group  $G_1$  and, ultimately, perhaps the (unrestricted) group  $\mathcal{M}_2$  suggested by the lattice of Figure 1, one may also wish to consider the successive construction of  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots$  or the corresponding restricted groups. Bart, for instance, has suggested that a person's capacity for combining propositions may be measured by the number  $n$  of generators of  $FB(n)$ , or more adequately, by the number of generators of the group operating on  $FB(n)$ . Bart's construction of that group will be discussed in Appendix 3. If we take that group to be  $\mathcal{M}_n$ , the number of generators increases as  $2^n$  so that, when progressing from  $n$  to  $n+1$ , we have to add  $2^n$  new generators. The same occurs in the case of Bart's group  $\Phi_n$ , although the structure of his groups is quite different. In appendix 2, we present Polya's construction of the restricted group of automorphisms of  $FB(n)$ . The situation improves radically. The number of generators is  $n$ ; at each step, we have to add one more generator.

### Relations and their Structure Preserving Transformations

We have so far relied on the argument of limited mobility for considering as psychologically relevant some subgroups only of the group  $\mathcal{M}_2$  of mixed automorphisms of  $FB(2)$ . But Piaget gives the clue for another course leading to a reduction, even to a more radical one.

Indeed, Piaget (1972, p. 339) says (but many other places may be found) "... that the reciprocity  $R$  corresponds to the inverse operation of the additive groupment of relations (and to the permutation of the terms of the equivalence or of the implication between propositions)". This prompts one to examine structure-preserving transformations of relations.

Now, Piaget asserts also: "The binary operations include only one transformation by permutation  $(p \supset q) \xrightarrow{P} (q \supset p)$  and vice versa, equivalent to the reciprocity  $(R)$ , namely  $q \supset p = R(P \supset q)$ " (Piaget, 1952, p. 48).

Without any further restriction, there are, of course, 14! such transformations. If, however, we take into account only automorphisms of  $FB(2)$ , then Table 2 shows that there are two of them which transform  $p \supset q$  into  $p \supset q$  and vice versa, that is,  $c$  into  $b$ , namely Piaget's  $R$  and the automorphism which we have labelled  $T$ . There is no  $d$ -automorphism that does the job. In fact, it is  $T$  that corresponds to the operation of conversion of a relation. (I prefer to use, as does TARSKI, "converse" instead of "inverse", because, in general,

the converse of a relation is not an inverse as understood in group theory). Indeed, under conversion, symmetric binary relations remain unchanged ; they are "self-converse". Since the automorphism that fixes the eight symmetric elements of  $FB(2)$ , namely  $\{v, a, b, e, j, k, o\}$ , is  $T$ , it will turn out that  $T$  is the transformation which accomplishes the conversion of relations. Presently, we shall elaborate this assertion and qualify it. But before, we have to recall the algebraic structure of relations on a set of elements. Detailed information may be found, for instance, in Tarski (1941) and in Jonnson and Tarski (1951 and 1952).

On the set  $A$  of binary relation it is possible to define Boolean (or absolute) operations : two binary operations  $\wedge$  and  $\vee$  (meet and join), one unary operation ' (complement) and two special relations  $0$  and  $1$  (null relation and universal relation) such that the algebra

$$\mathcal{B} = \langle A, \vee, ', 0, 1 \rangle$$

is a Boolean algebra, the "Boolean algebra of relations". The binary relations on a set  $A$  of  $n$  elements may be interpreted as subsets of the set  $A \times A$  with  $m^2$  elements. (Finer distinctions are possible and necessary in other contexts.) Then the operations meet, join and complement of  $\mathcal{B}$  simply are the set-theoretic operations of intersection, union and complement. Thus, the algebra  $\mathcal{B}(m)$  of binary relations defined on a set of  $m$  elements is isomorphic to a Boolean algebra  $B(m^2)$  with  $2^q$  ( $q = m^2$ ) elements. On the other hand, the free Boolean algebra  $FB(n)$  of  $n$ -ary propositional operations is, as we know

isomorphic to  $B(2^n)$ . Putting  $n = 2^p$ , we find the cases in which the Boolean algebra of relations is isomorphic to a free Boolean algebra (of propositional operations) :

$$\mathcal{B}(2^p) \simeq BL(2p), \quad (37)$$

i.e. the Boolean algebra of relations on a set of  $2^p$  elements is isomorphic to the free Boolean algebra with  $2p$  generators. Thus, the relations  $\mathcal{B}(2)$  on a set of two elements are isomorphic to  $FB(2)$  ;  $\mathcal{B}(3)$  is not isomorphic to any free Boolean algebra, in particular not to  $BL(3)$ , but  $\mathcal{B}(4)$  is isomorphic to  $BL(4)$ ,  $\mathcal{B}(8)$  to  $BL(6)$  and so on.

So far, the -Boolean- algebra of relations has not provided us with any means of reducing the number of automorphisms (and  $d$ -automorphisms) which should be taken into account. However, others than the Boolean operations can be defined on the set of binary relations.

These operations are termed Peircean or relative operations by Tarski. There are two binary operations, the multiplication  $\cdot$  and the addition  $+$ , one unary operation, the conversion  $\smile$ , and two special relations, the identity relation  $E$  and the diversity relation  $J$ , in addition to the relations  $0$  and  $1$  of the Boolean structure that play a role also in the Peircean structure. More precisely, we can say that this structure, the Peircean algebra of relations is given by

$$\mathcal{F} = \langle A, \cdot, +, \smile, ', E, J, 0, 1 \rangle,$$

where the operations have the special properties indicated below.

(i)  $A = \langle A, \cdot, E, 0 \rangle$  is a monoid (unit element  $E$ ) with null element  $0$ .

(ii)  $A_+ = \langle A, +, J, 1 \rangle$  is a monoid (unit element  $J$ ) with null element  $1$ .

(Note : addition, like multiplication, of relations is (not commutative !)

(iii) The unary operation  $\checkmark$  is characterized by :

$$(X)\checkmark = X, \tag{38}$$

$$(X \cdot Y)\checkmark = \checkmark Y \cdot \checkmark X, \tag{39}$$

$$(X+Y) = \checkmark \checkmark Y + \checkmark \checkmark X, \tag{40}$$

i.e.  $\checkmark$  is an idempotent anti-automorphism (a-automorphism) of each of the monoids  $A$  and  $A_+$ . ("Anti-", because the order of the elements is inversed.)

(iv) The unary operation  $'$ , in addition to its Boolean properties, obeys identities of the De Morgan type :

$$(X \cdot Y)' = X' + Y', \tag{41}$$

$$(X + Y)' = X' \cdot Y', \tag{42}$$

i.e.  $'$  is an isomorphism of the monoids  $A$  and  $A_+$ . By analogy with the Boolean case, we say also that  $'$  is a d-automorphism of

$\mathcal{F}$ . Moreover, the Boolean and the Peircean structures are linked by the distributivity (right and left) of  $\cdot$  with respect to  $\vee$  and by the distributivity (right and left) of  $+$  with respect to  $\wedge$

Again, if a relation is interpreted as a subset, then the following set-theoretic interpretation of the Peircean operations can be given :

-  $(y, z) \in X \cdot Y$  means that there exists an element  $u$  such that  $(y, u) \in X$  and  $(u, z) \in Y$  ;

-  $(y, z) \in x + y$  means that for all  $u$  one has either  $(y, u) \in X$  or  $(u, z) \in Y$  or both ;

-  $(y, z) \in \checkmark X$  means that  $(z, y) \in X$ .

Furthermore :

-  $(y, z) \in X'$  means the we have not  $(y, z) \in X$ .

We write from now on  $N$  for  $'$ . The next step is to establish carefully that one can write  $T$  for  $\checkmark$ .

We now substantiate and make more precise our claim that the converse of a relation  $X$  is the relation  $TX$  and that it cannot be  $PX$ . We shall first say what is meant by symmetric BPO. A BPO, i.e. an element of  $FB(2)$ , is symmetric if it is left unchanged by the exchange of the two generators chosen to express the elements of  $FB(2)$ . This definition obviously depends on the choice of generators - and not only on what we have termed generator system. Thus, for what may justifiably be called, the "standard generators", viz.  $(p, q)$  in that order, the eight symmetric BPO are  $\{v, a, d, e, j, k, x, o\}$ . the "standard symmetric BPO". It is the automorphism  $T$  that leaves this set invariant elementwise. These are the symmetric BPO also for the generators  $(q, p)$ ,  $(p', q')$  and  $(q', p')$ . The choices  $(p, q')$ ,  $(q', p)$ ,  $(p', q)$ , however, single out the BPO  $\{v, b, c, e, j, l, m, o\}$  as symmetric and the automorphism  $TR$  fixes each of these. Similarly, the two other generator systems give rise to two sets of symmetric BPO. (Cf. columns 4 and 5 of Table 1). This becomes almost obvious in the crystallographic representation of Appendix 1.

As we apply the transformations of  $\mathcal{M}_2$  to the 16 BPO, the standard generators successively transformed into each of the 24 possible generator pairs, the set of standard symmetric BPO is successively transformed into each of the six sets of symmetric BPO, and the subgroups that fix the symmetric BPO are transformed at the same time. If a transformation  $\tau \in \mathcal{M}_2$  transforms a symmetric BPO, say  $s$  into  $\tau s$  and if the transformation  $\phi$  fixes  $s$  ( $\phi s = s$ ), then; obviously,  $\tau \phi \tau^{-1}$  (the conjugate of  $\phi$  by  $\tau$ ) fixes  $\tau s$ . Thus, as  $\tau$  varies over the group  $\mathcal{M}_2$ , the automorphism  $T$  that fixes the standard symmetric BPO, is transformed successively into all elements of its conjugate class. Tables 2 and 3 are divided into conjugate classes. As can be seen, Piaget's  $R$  does not belong to the conjugate class of  $T$ . Thus, it cannot be, for any choice of generators, the transformation that fixes the symmetric BPO and transforms the others into their symmetric counterpart.

There remains to transfer these findings from the free Boolean algebra of the 16 BPO to the isomorphic algebra of binary relations on a set of two elements. To this end, we remark that the choice of an ordered pair of generators  $(p,q)$  in our case - enables us to regard the elements of  $FB(2)$  as Boolean functions. Let  $2$  be the set  $\{1,0\}$  and  $2_n$  this set ordered by  $0 \leq 1$ . A Boolean functions is a function on the  $n$ -fold direct product  $2 \times \dots \times 2$  (domain) to  $2_n$

(codomain)

$$\phi: 2 \times \dots \times 2 \rightarrow 2_n$$

We write  $2_n^{2 \dots 2}$  (with  $n$  times 2 in the exponent) for the set of all  $n$ -any Boolean functions.

It is now possible to identify elements of  $BL(n)$  with such functions. For instance  $a \in BL(2)$  is identified with the function that has the value 0 at the ordered pair  $(0,0)$  and the value 1 for the other three pairs (because  $p \vee q$  is false when both  $p$  and  $q$  are false, and true in the other three cases). Similarly, the value of  $m \in 2_n^{22}$  at  $(0,1)$  is 1 and it is 0 at the other arguments. This establishes an isomorphism

$$BL(2) \cong 2_n^{22} \tag{43}$$

We shall use the same symbols for the elements of  $BL(2)$  and the corresponding ones of  $2_n^{22}$ .

In the preceding section, we had chosen a generator system and then found the invariance group  $G_1$  of that system. It is clear that essentially, what we have found there is the group of mixed automorphisms of the Boolean functions  $2_n^{22}$ . The subgroup of automorphisms is isomorphic to the automorphisms of the domain of these functions, i.e. of the direct product  $2 \times 2$ ; this group is isomorphic to  $D_4$ , the  $d$ -automorphisms are obtained by adjoining to these the (only)  $d$ -automorphism

of the codomain  $\mathcal{B}$ .

Remember, however, that the isomorphism (43) is based on the choice of a definite pair of generators : the standard pair (p,q). It may, therefore, be termed the 'standard isomorphism'. But if, for instance, we choose the generators (p,e), then it would be the element 'pvq' that corresponds to the Boolean function a, and 'p'∧'q' would correspond to m.

Exactly in the same way, i.e. by a choice of generators of  $FB(n)$ , it is possible to establish an isomorphism

$$BL(n) \approx \mathcal{B}^{2 \dots 2} \quad (44)$$

The next step is to provide an isomorphism

$$\mathcal{B}^{22} \approx \mathcal{B}(2) \quad (45)$$

A convenient way is to make correspond to a Boolean function  $\phi \in \mathcal{B}^{22}$  the relation  $\phi$  defined by

$$(y,z) \in \phi \text{ if and only if } \phi(y,z) = 1$$

(i.e. to consider the Boolean function as the characteristic function of a subset). This establishes a one-to-one mapping between  $\mathcal{B}^{22}$  and  $\mathcal{B}(2)$ . We shall use the same symbols for the elements thus related. By the same token, the automorphisms of  $FB(2)$  now represent isomorphisms between  $FB(2)$  and  $\mathcal{B}(2)$  ; the standard isomorphism corresponds to the identity automorphism.

We have defined what are meant by symmetric BPO. It is clear that this definition carries over to binary Boolean functions and to binary relations. A binary Boolean function  $\phi$  is symmetric if, for any  $y, z \in \mathcal{B}$

$$\phi(z,y) = \phi(y,z).$$

A binary relation  $\phi$  is symmetric if, for any  $y, z,$

$$(z,y) \in \phi \text{ implies } (y,z) \in \phi.$$

This shows : (i) that, with the standard isomorphism, it is  $T$  that leaves symmetric relations invariant, and (ii) that the operation  $\checkmark$  may be identified with the transformation  $T$  :

$$TX = \checkmark X \quad (46)$$

Concerning the role of automorphisms, our previous discussion of symmetric BPO carries over too. In any case, the transformation that leaves invariant a symmetric relation, must belong to the conjugate class of  $T$  ; therefore, it cannot be  $R$ . Furthermore - from (39), (40) and (46) - we know already that  $T$  is not an automorphism of the Peircean structure ; it is an anti-automorphism of  $\mathcal{F}(2)$ . Which, then, of the Boolean automorphisms are also Peircean ones ?

The only automorphisms with respect to the Peircean structure of relations on a finite set come from a renumbering of the elements of the set. Thus, the Peircean automorphisms

of  $\mathcal{F}(n)$  form the group  $S_n$  of all permutations of  $n$  objects. For  $\mathcal{F}(2)$ , this is the group of the order two ; the automorphism itself is easily recognized as Piaget's  $R$ , in case the standard isomorphism is chosen. For any choice of isomorphism, this automorphism is an element of the conjugate class of  $R$ , which is  $\{P, Q, R\}$ . Thus,  $R$  does not correspond to the "inverse operation of relations", but is the only Peircean automorphism of  $\mathcal{F}(2)$  - in addition, of course, to the identity.

If we take into account the Boolean and the Peircean structure of relations, we have

$$\mathcal{R} = \langle A, \wedge, \vee, \cdot, +, N, T, 0, 1, E, J \rangle$$

the "algebra of relations". We are interested in the structure preserving transformations of  $\mathcal{R}$ . We have met four types of transformation.

(i) Transformations that are automorphisms of  $\mathcal{B}$  and of  $\mathcal{F}$ .

We have seen that  $R$  is an automorphism of  $\mathcal{B}(2)$  and of  $\mathcal{F}(2)$ . Such transformations will be termed automorphisms of  $\mathcal{R}$ .

(ii) Transformations that are d-automorphisms of  $\mathcal{B}$  and of  $\mathcal{F}$ , for instance  $N$  and  $RN = \bar{R}$  are d-automorphisms of  $\mathcal{B}(2)$  and of  $\mathcal{F}(2)$ . There are the d-automorphisms of  $\mathcal{R}$ .

(iii) Transformations that are automorphisms of  $\mathcal{B}$  and a-automorphisms of  $\mathcal{F}$ . Since the binary operations of  $\mathcal{B}$  are commutative, there are no a-automorphisms of  $\mathcal{B}$ . We

call these the a-automorphisms of  $\mathcal{R}$ . Example :  $T$  and  $TR$ .

(iv) Transformations that are d-automorphisms of  $\mathcal{B}$  but are dual a-automorphisms with respect to  $\mathcal{F}$ . These are the da-automorphisms of  $\mathcal{R}$ . The transformations  $TN$  and  $T\bar{R}$  are da-automorphisms of  $\mathcal{R}(2)$  :

$$TN(r.s) = TNs + TNr \quad (47)$$

It is easily seen that  $T$  and  $N$  - in addition to commuting one with the other - also commute with all automorphisms of

$\mathcal{F}(n)$ . Furthermore,  $T$  and  $N$  cannot be obtained by any renumbering. Thus, the group  $\mathcal{G}_n$  of all automorphisms, d-automorphisms, a-automorphisms and da-automorphisms of  $\mathcal{R}(n)$  is a direct product

$$\mathcal{G}_n = S_n \times \langle T \rangle \times \langle N \rangle \simeq S_n \times D_2, \quad (48)$$

where  $D_2$  is generated by  $T$  and  $N$ . In particular

$$\mathcal{G}_2 = \langle R \rangle \times \langle T \rangle \times \langle N \rangle, \quad (49)$$

a group we have already met as  $G_2$  in the preceding section (on subgroups).

Any subgroup  $L$  of  $\mathcal{G}_n$  that contains automorphisms, a-automorphisms, d-automorphisms and da-automorphisms, contains the same number of each sort. The order of such a group is a multiple of four. In general, however, such groups are not direct products of the subgroup of automorphisms  $L'$  with  $D_2$  ; but

always

$$L/L' \cong D_2 \quad (50)$$

(i.e.  $L$  is an extension of  $L'$  with  $D_2$ ). Of course, also

$$\mathfrak{G}_n/S_n \cong D_2 \quad (51)$$

In conclusion, a group isomorphic to  $D_2$  does appear in quite an essential way whenever one considers structure preserving transformations of algebras of relations. This group, however, is not INRC but  $\{I, T; N, TN\}$ .

#### Concluding Remarks

We have discussed in some detail some of the questions concerning transformations of propositional operations. It is, of course, impossible to be complete within the limits of a paper, even in such a limited field. But whatever the coverage may be, care of details is necessary and rewarding; the link with experiment is at this level and at this price. Here we were specially interested in the gradual building-up of groups of structure preserving transformations from involutions, and in the manner transformations other than involutions come about through this process. The paper should also illustrate the kind of reflection that could be important in modelisation of the algebra of competence.

#### Appendix 1

##### Crystallographic Representations of $\mathcal{M}_n$

The group  $\mathcal{M}_n$  of mixed automorphisms of the free Boolean algebra with  $n$  generators  $FB(n)$  is isomorphic to the symmetry group of a  $(2^n-1)$ -dimensional crystal. In the case of  $FB(2)$ , the Boolean algebra of the sixteen BPO, the crystal is an ordinary, that is three-dimensional, crystal. The group  $\mathcal{M}_2$  is then the group of all spatial symmetries of a cube. (In crystallographic notation  $\mathcal{M}_2 \cong m\bar{3}m$ ,  $\mathcal{A}_2 \cong \bar{4}3m$ ).

Therefore, it is easy to represent the elements of  $\mathcal{M}_2$  by spatial transformations. We only need to place the elements of  $FB(2)$  suitably on the eight vertices and the six centers of the faces of a cube, e.g. as shown in Figure 2. This is the configuration of the face-centered cube, well known to crystallographers. Two of the 16 BPO are missing, viz. the tautology  $v$  and the antilogy  $o$ . These elements undergo only two transformations: either they remain unchanged (under automorphisms) or they are exchanged (by d-automorphism).

To each element of  $\mathcal{M}_2$  there corresponds now a symmetry operation of the cube, as shown in Table 5. The elements of  $\mathcal{M}_2$  are given by the products found in the column 'Generators' of Tables 2 and 3, the transformations of the cube by their international crystallographic symbol to which subscripts have been added that specify their orientation. The position of

these symmetry elements with respect to the cube, is shown in Figure 3 ; the meaning of the symbols is explained in the following remarks.

Let us first observe that the transformation  $N$  (already termed "inverse" by Piaget) corresponds to inversion  $\bar{1}$  through the centre  $O$  of the cube. All axes of Figure 3 contain  $O$ . Three of them,  $x, y,$  and  $z,$  pass through the centers of opposite faces. About these axes there are rotations by  $180^\circ$  (such as  $2_x$ ), rotations by  $90^\circ$  ( $4_x$ ) or by  $270^\circ$  ( $4_x^3$ ). Furthermore, these transformations combined with the inversion  $\bar{1}$  also occur. The combination  $2_x\bar{1}$  is shortened as  $m_x$ ; it represents reflexion through a plane, a mirror, perpendicular to the x-axis (and passing through  $O$ ). The combination  $4_x\bar{1}$  is written  $\bar{4}_x$  and is termed a rotatory inversion.

Four axes,  $\alpha, \beta, \gamma$  and  $\delta,$  pass through the vertices of the cube. Around these axes there are rotations by  $120^\circ$  ( $3_\delta$ ) and by  $240^\circ$  ( $3_\delta^2$ ) as well as the corresponding rotatory inversions ( $\bar{3}_\delta, 3_\delta^2 = \bar{3}_\delta^5$ ).

Positive and negative rotations about these two families of axes are distinguished ; therefore they are fitted with one arrowhead. There are, however, six axes,  $A, B, C, D, E$  and  $F,$  fitted with two arrowheads around which only rotations of  $180^\circ$  ( $2_A$ ) and reflexions ( $m_A$ ) through planes perpendicular to them (and passing through  $O$ ) occur.

Remember now that the standard symmetric BPO are  $\{v, a, d, e, j, k, x, o\}$ . Except for  $v$  and  $o,$  which are not represent-

ed in the figure, they are situated in the plane corresponding to  $m_B,$  and, of course, they are left unchanged by reflexion in that plane, that is by  $T = m_B.$  The symmetric BPO with respect to the generators  $(p, q')$  are located in the plane corresponding to  $m_A$  and are fixed by  $m_A = TR.$  These planes intersect in the x-axis (the A-axis and the B-axis are perpendicular to the x-axis). The x-axis  $O_x$  is distinctive of the generator system  $\{p, q\}$  to which the generators  $(p, q)$  and  $(p, q')$  belong. The group  $G_1$  preserving this system, detailed in Table 4, singles out the x-axis ; it is the symmetry-group of a square prism, with the square perpendicular to the x-axis ( $G_1 \approx 4_x/mmm, G_1' \approx \bar{4}_x 2_y m$ ). The other two generator systems correspond to the other two directions parallel to the edges of the cube :  $\{i, e\}$  to  $O_y$  and  $\{e, h\}$  to  $O_z.$

Many other features discussed in the paper may be read off from the Figures 2 and 3. We cannot detail them all. Let us just mention the following. Not only the action of an element  $\tau$  of  $\mathcal{M}_2$  on the BPO may easily be found from Figure 1, but also the conjugate by  $\tau$  of an element  $\phi$  of  $\mathcal{M}_2,$  that is  $\phi \tau \phi^{-1}.$  It is obtained by performing on Figure 3 the geometric transformation corresponding to  $\tau.$  For instance,  $3_\delta m_C 3_\delta^{-1} = m_E$  and indeed performing a rotation of  $120^\circ$  around the axis  $\delta$  takes the C-axis to the place of the E-axis. This is easier to do than to compute directly  $(TU)(UP)(UT).$

Appendix 2

More about the "Restricted Groups"

The group  $G_1'$ , the restricted group of automorphisms of the section on subgroups, a group isomorphic to  $D_4$ , has already been found by Pólya (1937 and 1940). He posed and solved a logical problem whose origin he traces to Jevons. The first step towards the solution, the only one in which we are interested here, is to find the "hypercubic" group  $K_n$  formed by the operations of permuting and/or complementing the  $n$  variables of  $n$ -any Boolean functions. The  $2^n$  possible values of the argument of such a function can be placed on the vertices of a hypercube, that is a cube in  $n$  dimensions. For  $n = 2$  this is a square ; for  $n = 3$  it is the ordinary cube as shown in Figure 4. Note that this utilisation of the cube is different from that in Appendix 1.

The structure of  $K_n$  has been determined by Polya as "wreath product" of the group of two elements,  $C_2$  by the group  $S_n$  of all permutations of  $n$  objects. This wreath product is a semi-direct product of  $S \times \dots \times C$  ( $n$ -times) by  $S_n$

$$(C_2 \times \dots \times C_2) \dot{\times} S_n \quad (52)$$

where an element  $s$  of  $s_n$  acts on  $C_2 \times \dots \times C_2$  by permuting the  $n$  generators of the  $n$  copies of  $C_2$ . Therefore, Pólya (1937) uses the pictorial expression " $C_2$  - wreath around  $S_n$ ".

The group  $K_n$  has  $2^n n!$  elements. For unary, binary and ternary Boolean functions we have the following isomorphisms

$$K_1 \cong C_2 \quad (53)$$

$$K_2 \cong D_4 \cong G_1' \quad (54)$$

$$K_3 \cong S_4 \times C_2 \quad (55)$$

Thus  $K_3$ , the group of automorphism of the ternary Boolean functions  $2^{2 \times 2 \times 2}$ , which is also the restricted group of automorphisms of  $FB(3)$ , is isomorphic to the group of all mixed automorphisms of the binary propositional operations  $FB(2)$ . Therefore, we should be able to read the transformations of Tables 2 and 3 as giving all transformations of commuting and/or complementing the three standard generators, say  $p, q$  and  $r$ , of  $FB(3)$ . (There are of course many other choices). This is indeed easily done. We restrict the transformations in question to the set  $\{i, f, h, g, e, j\}$  and we identify these elements with  $p, q', q, q', r$  and  $r'$  respectively. Note, however, that in our former interpretation as elements of  $FB(2)$ , the elements  $e$  and  $j$  were not independent of  $i, f, h$  and  $g$  ; now as generators of  $FB(3)$ , they are independent, of course. Placed at the centres of the faces of the cube (as in Figure 4), they form a three-dimensional octahedron ; in general we find a  $n$ -dimensional octahedron.

Let us now examine how the restricted groups may be successively built up as we add one generator at a time to a

free Boolean algebra. For  $n = 1$  we have as single generator of  $K_1 = C_2$  the permutation  $(pp')$  which we note  $\bar{P}$ :

$$\bar{P} = (pp') \quad (56)$$

(Inspection of Tables 2 and 3 shows that there are eight elements of  $\mathcal{M}_2$  which contain  $(pp') = (if)$ . The reason for the choice of  $\bar{P}$  will become clear, presently). For  $n = 2$  we add as second generator of

$$K_2 = (C_2 \times C_2) \dot{\times} S_2$$

the element

$$T = (pq) (p'q') \quad (57)$$

the generator of  $S_2$ ; it permutes the two selected generations of  $FB(2)$ . (In fact there are two elements of  $\mathcal{M}_2$  that contain these permutations). We then find

$$\bar{TPT} = \bar{Q} = (qq'). \quad (58)$$

For

$$K_3 = (C_2 \times C_2 \times C_2) \dot{\times} S_3$$

we have to add a generator that, together with  $T$  generates  $S_3$ . We can take

$$U = (pr) (p'r'). \quad (59)$$

Then

$$\bar{UPU} = \bar{R} = (rr'). \quad (60)$$

And so on for higher values of  $n$ . Thus the number of generators of the group  $K_n$  of restricted automorphisms of  $FB(n)$  is  $n$ .

Incidentally, we have thus generated, (as mentioned in the section on dual automorphisms)  $\mathcal{M}_2 \approx S_4 \times C_2$  by the three involutions  $\bar{P}, T$  and  $U$ . However, thus generated, some apparently simple operations get complicated expressions. For instance

$$N = \bar{P}\bar{T}\bar{P}\bar{U}\bar{P}\bar{U}\bar{T}. \quad (61)$$

such complicated generation should be avoided, unless it reflects genetical complexities.

More recently, Leresche (1978) set out to determine "the complete group of binary [propositional] operations". By binary operations he understands the 10 (out of 16) binary operations, that actually depend on two arguments,  $p$  and  $q$ . In our notation, these are the 10 elements of the set

$$\{a, b, c, d, e, j, k, l, m, x\} = L.$$

From the preceding it is not quite clear, however, what kind of transformations should be selected out of the 10! candidates since the set  $L$  has no structure.

Indeed, it is not closed under meet and/or join. If then, to remedy, one adds the elements  $v$  and  $o$ , the distributivity equations (9) and (10) do not hold in general, and complementation is not unique. The set in question, even augmented, is

not a Boolean algebra and, a fortiori, not a Boolean subalgebra of  $FB(2)$ .

The way out is to say that one is interested only in those transformations belonging to  $\mathcal{M}_2$  that transform the elements of  $L$  among themselves. Consequently, these transformations transform the remaining 6 elements of  $FB(2)$  among themselves. Since the elements  $v$  and  $o$  always keep apart, this amounts to looking for those transformations of  $\mathcal{M}_2$  that permute the four elements  $i, h, g$  and  $f$  among themselves. But this is exactly the program that in the section on subgroups led to the group  $G_1$  of Table 4, whose subgroup of automorphisms  $G_1'$  is the group  $K_2$  already found by Pólya. However, the "complete group of transformations" found by Leresch, in his otherwise correct paper, is only the Boolean subgroup  $G_2$  of  $G_1$ , whose elements are given in (35). What is missing is the transformation  $T$  that exchanges the two generators. Similarly, for ternary propositional operations, he finds a Boolean group of order 16 instead of  $K_3 \times C_2 \cong S_4 \times D_2$  which is of order 96.

### Appendix 3

#### Bart's Generalization Model

In an interesting paper, Bart (1971) has chosen another way to restrict the abundance of the symmetric group  $S_{16}$ . He requires two things of the transformation group :

(i) Every element of the group should be an involution. Such

groups are sometimes termed Boolean groups.

(ii) The group should be a regular permutation group.

A group  $G$  of permutations on a set  $S$  is transitive if for any  $x, y \in S$  there exists an element  $\alpha \in G$ , such that  $\alpha x = y$ . Such a group is regular, if no element of it, except, of course, the unit element leaves any elements of  $S$  unchanged. If  $S$  is finite, then the number of elements of a regular group  $G$  (the order of  $G$ ) equals the number of elements of  $S$  (the degree of  $G$ ). A commutative and transitive group of permutations is regular.

Taken separately, the two conditions are not very restrictive ; there are many subgroups of  $S_{16}$  that fulfill them. But taken together, do they seriously limit the possibilities ? Bart's procedure for obtaining his group seems rather natural. It may be formulated as follows :

(i) From the elements  $FB(2)$ , one forms a ring by defining additions by

$$x + y = xy, \tag{62}$$

and multiplication by  $xy = x \wedge y$ . (This is Stone's construction).

(ii) With this addition, the elements of  $FB(n)$  constitute a Boolean group  $H_n$  with unit element the zero  $0$  of  $FB(n)$ .

(This is why the group is termed Boolean.) Further it has by construction, the right number of elements, namely  $2^p$  ( $p = 2^n$ ).

(iii) A natural way to define the action  $\phi_\xi$  of an element  $\xi$  of  $H_n$  on an element  $x$  of  $FB(n)$  is to define

$$\phi_{\xi}(x) = \xi + x. \quad (63)$$

Due to the group properties of  $H_n$ , the set

$$\phi_n = \{\phi_{\xi} : \xi \in K_n\} \quad (64)$$

is a regular Boolean group of transformations of  $BL(n)$ , and

$$\phi_{\xi} \phi_{\eta} = \phi_{\xi + \eta}. \quad (65)$$

The dual construction of a Boolean ring by

$$x +' y = x \leftrightarrow y, \quad x \cdot y = x \vee y \quad (66)$$

gives rise to transformations  $\phi_{\xi}'$  defined by

$$\phi_{\xi}'(x) = x +' y. \quad (67)$$

The group obtained this way is again  $\phi_n$ , because

$$\phi_{\xi}' = \phi_{\xi'}, \quad (68)$$

where  $\xi'$  is the complement, in  $FB(n)$ , of  $\xi$ .

Bart's group  $\phi_2$  is shown in Table 6. It will be noticed that  $I = \phi_0$  and  $N = \phi_v$  are the only elements of  $\mathcal{M}_2$  that are also elements of  $\phi_2$ .

However, for  $n > 1$ ,  $\phi_n$  is not the only regular Boolean transformation group of the elements of  $BL(n)$ .  $\phi_n$  is not a normal subgroup of  $S_q$  where  $q$  ( $q = 2^p$ ,  $p = 2^n$ ) is the number of elements of  $BL(n)$ ; the elements of  $\phi_n$  do not form a complete conjugate class of  $S_q$ . (For  $n = 2$ , they

belong to a class with roughly two million elements). All the (numerous) subgroups of  $S_q$  conjugate to  $\phi_n$  are regular Boolean transformation groups of the elements of  $FB(n)$ . (Furthermore, they arise from commutative binary operations on the set of elements of  $BL(n)$  in the same way  $\phi_n$  arises from the operation  $w$  in (64).

We have now shown how to obtain a great number of regular Boolean transformations on the set of  $n$ -ary propositional operations. It has, however, been argued in the text, that if we model reversibility in cognitive activity operations  $\alpha$  having an inverse  $\alpha^{-1}$  (as understood in group theory), the further requirement that  $\alpha^{-1} = \alpha$  termed "strong view of reversibility" (Bart, 1971, p. 542) is too strong. If involutions play a role, then the experience of their possible non-commutativity leading to higher order operations is necessary for later stages of cognitive activity. As regards transitivity, Piaget was conscious of the fact that INRC is not transitive, and was not at all bothered by it. The logical meaning of regular Boolean transformations on the set of elements of  $n$ -ary propositional operations is not too clear. Meet, join and complement are closely connected with the fact that the elements in question are propositional operations, the transformations of  $\phi_n$  do not respect these operations. The price paid for regularity and involutive character seems too high.

## References

- Bart, W. B. A generalization of Piaget's logical-mathematical model for the stage of formal operations.  
Journal of Mathematical Psychology, 1971, 5, 539-553
- Beth, E. W. A propos d'un "Traité de logique".  
methodos, 1950, 2, 258-264.
- Hoffman, W. C. Mathematical models of Piagetian Psychology.  
in S.Mogdil & C. Mogdil (Eds.) Towards a theory of psychological development.  
Windsor, Berks. : NFER Publishing Company, 1980.
- Jónsson, B. and Tarski, A. Boolean Algebra with Operators  
Parts I and II.  
American Journal of Mathematics, 1951, 73, 891-939 and  
1952, 74, 127-167.
- Leresche, G. INRC et groupes de transformations des opérations logiques.  
Revue Européenne des Sciences Sociales, 1976, 14, 219-241.
- Piaget, J. Traité de logique. Essai de logistique opératoire  
Paris : A. Colin, 1949.
- Piaget, J. La réversibilité de la pensée et les opérations logiques.  
Bulletin de la Société française de Philosophie,  
1950, 44, 137-156.
- Piaget, J. Essai sur les transformations des opérations logiques  
Paris : Presses universitaires de France, 1952.

- Piaget, J. Logique formelle et psychologie génétique.  
In P. Fraise, F. Bresson & J.-M. Favarge (Eds.), Les modèles et la formalisation du comportement. Paris : Editions du Centre National de Recherche Scientifique, 1967. Pp. 269-276.
- Piaget, J. Genetic Epistemology.  
New-York : Columbia University Press, 1970.
- Piaget, J. Essai de Logique Opératoire.  
2<sup>d</sup> edition of Piaget (1949) set up by J.-B. Grize.  
Paris : Dunod, 1972.
- Pólya, G. Kombinatorische Anzahlbestimmung für Gruppen, Graphen und chemische Verbindungen.  
Acta mathematica, 1937, 68, 145-254.
- Pólya, G. Sur les types des propositions composées.  
The Journal of Symbolic Logic, 1940, 5, 98-103.
- Tarski, A. On the calculus of relations.  
The Journal of Symbolic Logic, 1941, 6, 73-89.
- Wermus, H. Les transformations involutives (réciprocités) des propositions logiques. Archives de Psychologie, 1971, 41, 153-171.

## Footnotes

- This work was supported by the Fonds National Suisse de la recherche scientifique (project 1.345-0.81).
- I have profited from discussion with Gil Henriques and I enjoy the hospitality of the Department of Theoretical Physics of the University of Geneva, Switzerland.

Table 1

Notations and Generators for the Free Boolean Algebra FB(2)

	(i,h)		(i,e)	(e,h)
v	1	{k,l,m,x}	1	1
a	$p \vee q$	{k,l,m}	$i \vee e'$	$e' \vee h$
b	$p \vee q'$	{k,l,x}	$i \vee e$	$e \vee h'$
c	$p' \vee q$	{k,m,x}	$i' \vee e$	$e \vee h$
d	$p' \vee q'$	{l,m,x}	$i' \vee e'$	$e' \vee h'$
i	p	{k,l}	i	$e \leftrightarrow h$
h	q	{k,m}	$i \leftrightarrow e$	h
e	$p \leftrightarrow q^a$	{k,x}	e	e
j	$p \wedge q^b$	{l,m}	$e'$	$e'$
g	$q'$	{l,x}	$i \wedge e$	$h'$
f	$p'$	{m,x}	$i'$	$e \wedge h$
k	$p \wedge q$	{k}	$i \wedge e$	$e \wedge h$
l	$p \wedge q'$	{l}	$i \wedge e'$	$e' \wedge h'$
m	$p' \wedge q$	{m}	$i' \wedge e$	$e' \wedge h$
x	$p' \wedge q'$	{x}	$i' \wedge e'$	$e \wedge h'$
o	0		0	0

Note : For columns 4 and 5 see under the heading 'Relations'.

$$^a p \leftrightarrow q = (p \wedge q) \vee (p' \wedge q')$$

$$^b p \wedge q = (p \wedge q') \vee (p' \wedge q)$$

Table 2

Group  $\mathcal{A}_2$  of Automorphisms of FB(2)

$S_4$	Extended to $S_{16}$	Generators
(k)(l)(m)(x)	(a)(b)(c)(d)(i)(h)(e)(j)(g)(f)I	
(mx)(k)(l)	(ab)(c)(d)(he)(jg)(i)(f)	UTU = VP
(lx)(k)(m)	(ac)(b)(d)(ie)(jf)(h)(g)	U
(kx)(l)(m)	(ad)(b)(c)(ig)(hf)(e)(j)	VUV = TR
(lm)(k)(x)	(bc)(a)(d)(ih)(gf)(e)(j)	T
(km)(l)(x)	(bd)(a)(c)(ij)(ef)(h)(g)	TVT = UQ
(kl)(m)(x)	(cd)(a)(b)(hj)(eg)(i)(f)	V
(kl)(mx)	(ab)(cd)(hg)(ej)(i)(f)	UTUV = P
(km)(lx)	(ac)(bd)(if)(ej)(h)(g)	VUVT = Q
(kx)(lm)	(ad)(bc)(if)(hg)(e)(j)	TVTU = R
(lxm)(k)	(abd)(d)(ieh)(jgf)	TU
(kxm)(l)	(abd)(c)(igj)(hef)	TUVT
(lmx)(k)	(acb)(d)(ihe)(jgf)	UT
(lkx)(m)	(acd)(b)(ieg)(jhf)	UV
(kmx)(l)	(adb)(c)(ijg)(ehf)	TVUT
(lkx)(m)	(adc)(b)(ige)(jfh)	VU
(lkm)(x)	(bcd)(a)(ihj)(efg)	TV
(lmk)(x)	(bdc)(a)(ijh)(egf)	VT
(lkxm)	(abcd)(iefj)(hg)	TUV = UP = RU
(lxmk)	(abdc)(igfh)(ej)	VTU = TQ = PT
(lmkx)	(acbd)(jheg)(if)	UVT = VR = QV
(lkmx)	(acdb)(ihfg)(ej)	UTV = QT = TP
(lxkm)	(adbc)(jgeh)(if)	TVU = RV = VQ
(lmxk)	(adcb)(ijfe)(hg)	VUT = PU = UR

Table 3

Dual Automorphisms of  $FB(2)$

Elements of $S_{16}$	Generators
(ax) (bm) (cl) (dk) (if) (hg) (ej)	N
(am) (bx) (cl) (dk) (if) (hj) (eg)	VP
(al) (bm) (cx) (dk) (ij) (hg) (ef)	UN
(ak) (bm) (cl) (dx) (ih) (fg) (ej)	$\overline{TR}$
(ax) (bl) (cm) (dk) (ig) (hf) (ej)	TN
(ax) (bk) (cl) (dm) (ie) (jf) (hg)	$U\overline{Q}$
(ax) (bm) (ck) (dl) (if) (he) (jg)	VN
(am) (bx) (ck) (dl) (if) (h) (e) (j) (g)	$\overline{P}$
(al) (bx) (ck) (dm) (hg) (i) (e) (j) (f)	$\overline{Q}$
(ak) (bl) (cm) (dx) (ej) (i) (h) (g) (f)	$\overline{R}$
(amcxbl) (dk) (ijhfeg)	TUN
(amdxbk) (cl) (ihjfgc)	TUVTN
(albxcm) (dk) (igefhj)	UTN
(aldxck) (bm) (ijgfeh)	UVN
(akbxdm) (cl) (iegfjh)	TVUTN
(akcdkl) (bm) (ihefgj)	VUN
(bldmck) (ax) (igjfhe)	TVN
(bkcmdl) (ax) (iehfjg)	VTN
(amcx) (bldx) (ijfe) (h) (g)	$TUVN = \overline{RU}$
(amd) (bkcx) (ihfg) (e) (j)	$VTUN = \overline{PT}$
(albk) (cmdx) (ehjg) (i) (f)	$UVTN = \overline{QV}$
(aldm) (bxck) (igfh) (e) (j)	$UTVN = \overline{TP}$
(akbl) (cxdm) (egjh) (i) (f)	$TVUN = \overline{VQ}$
(akcm) (bkdl) (iefj) (h) (g)	$VUTN = \overline{UR}$

Table 4

Automorphisms and d-Automorphisms that Preserve the Generator System  $\{p, q\}$

Permutations	Generators	Crystallographic interpretation
(p) (q) (p') (q')	I $\overline{R} = NC$	$1 \quad m_x$
(pp') (q) (q')	Q $\overline{P}$	$2_y \quad m_z$
(qq') (p) (p')	$QTQ = P \quad \overline{Q} = T\overline{P}T$	$2_z \quad m_y$
(pp') (qq')	$QTQT = R^a \quad N$	$2_x \quad \overline{1}$
(pq) (p'q')	T $\overline{TR}$	$m_B \quad 2_A$
(pq') (qp')	$QTQ = \overline{P}T\overline{P}^b \quad TN$	$m_A \quad 2_B$
(ppq'q')	QT $\overline{PT}$	$\overline{4}_x \quad 4_x^3$
(pp'p'q')	TQ $\overline{TP}$	$\overline{4}_x^3 \quad 4_x$

Note : The crystallographic interpretation is presented in Appendix 1.

<sup>a</sup>  $QTQT = PTPT = R,$

<sup>b</sup>  $QTQ = \overline{P}T\overline{P} = TR$

Table 5

Crystallographic interpretation of  $\mathcal{M}_2$  as  $m\bar{3}m$

$\mathcal{A}_2$	$\bar{4}3m$		
I	1	N	$\bar{1}$
VP	$m_F$	$VP$	$2_F$
U	$m_D$	UN	$2_D$
TR	$m_A$	$TR$	$2_A$
T	$m_B$	TN	$2_B$
UQ	$m_C$	$UQ$	$2_C$
V	$m_F$	VN	$2_F$
P	$2_z$	$\bar{P}$	$m_z$
Q	$2_y$	$\bar{Q}$	$m_y$
R	$2_x$	$\bar{R}$	$m_x$
TU	$3_\delta$	TUN	$\bar{3}_\delta$
TUVT	$3^2$	TUVTN	$\bar{3}_\gamma^5$
UT	$3_\delta^2$	UTN	$\bar{3}_\delta^5$
UV	$3_\beta$	UVN	$\bar{3}_\beta$
TVUT	$3_\gamma$	TVUTN	$\bar{3}_\gamma^5$
VU	$3_\beta^2$	VUN	$\bar{3}_\beta^5$
TV	$3_\alpha^2$	TVN	$\bar{3}_\alpha^5$
VT	$3_\alpha$	VTN	$\bar{3}_\alpha$
UP	$\bar{4}_y^3$	$\bar{R}U$	$4_y^3$
TQ	$\bar{4}_x^3$	$\bar{P}T$	$4_x^3$
VR	$\bar{4}_z$	$\bar{Q}V$	$4_z$
QT	$\bar{4}_x$	$\bar{T}P$	$4_x$
RV	$\bar{4}_z^3$	$\bar{V}Q$	$4_z^3$
PU	$\bar{4}_y$	$\bar{U}R$	$4_y$

Table 6

Bart's group of transformations of BL(2)

$\xi$	$\phi_\xi$
v	(vo)(ax)(bm)(cl)(dk)(if)(hg)(ej)
a	(vx)(ao)(bf)(cg)(de)(im)(hl)(jk)
b	(vm)(af)(bo)(cj)(dh)(ix)(el)(gk)
c	(vl)(ag)(bj)(co)(di)(hx)(em)(fk)
d	(vk)(ae)(bh)(ci)(do)(jx)(gm)(fl)
i	(vf)(am)(bx)(cd)(io)(hj)(eg)(kl)
h	(vg)(al)(bd)(cx)(ij)(ho)(ef)(km)
e	(vj)(ad)(bl)(cm)(ig)(hf)(eo)(kx)
j	(ve)(ak)(bc)(dx)(ih)(jo)(gf)(lm)
g	(vh)(ac)(bk)(dm)(ie)(jf)(go)(lx)
f	(vi)(ab)(ck)(dl)(he)(jg)(fo)(mx)
k	(vd)(aj)(bg)(cf)(il)(hm)(ex)(ko)
l	(vc)(ah)(be)(df)(ik)(jm)(gx)(lo)
m	(vb)(ai)(ce)(dg)(hk)(jl)(fx)(mo)
x	(va)(bi)(ch)(dj)(ek)(gl)(fm)(xo)
o	(v)(a)(b)(c)(d)(e)(f)(g)(h)(i)(j)(k)(l)(m)(n)(o)

Figure captions

Figure 1. Lattice of subgroups of  $\mathcal{M}_2$ .

Figure 2. Cubic symmetry of binary propositional operations.

Figure 3. Symmetry elements of the cube.

Figure 4. The restricted group  $K_3$  of automorphisms of  $BL(3)$ .

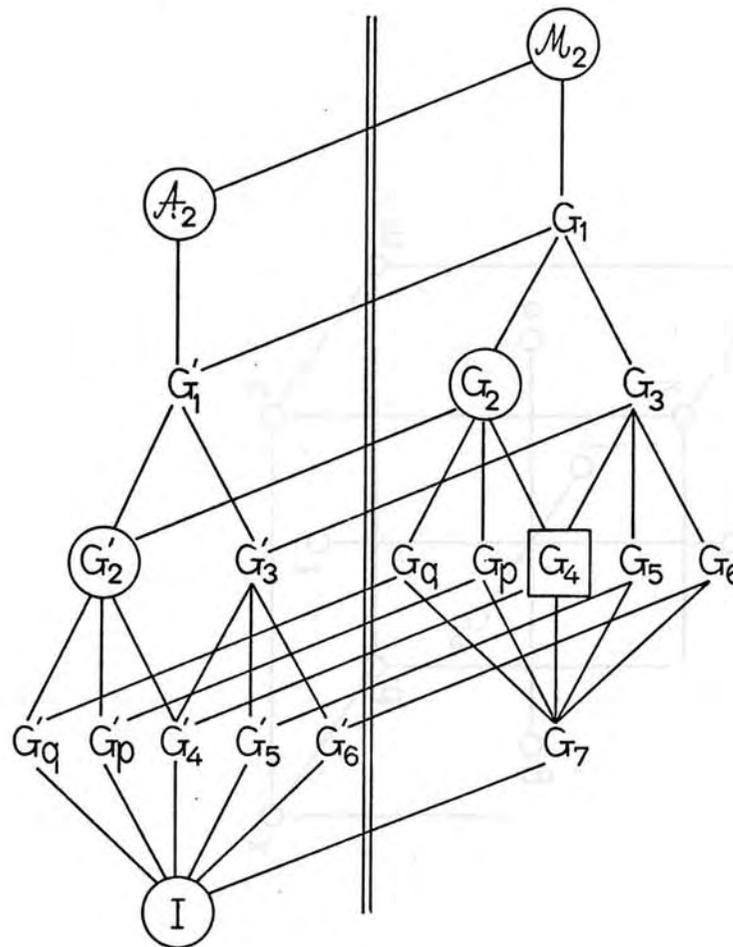


Figure 1

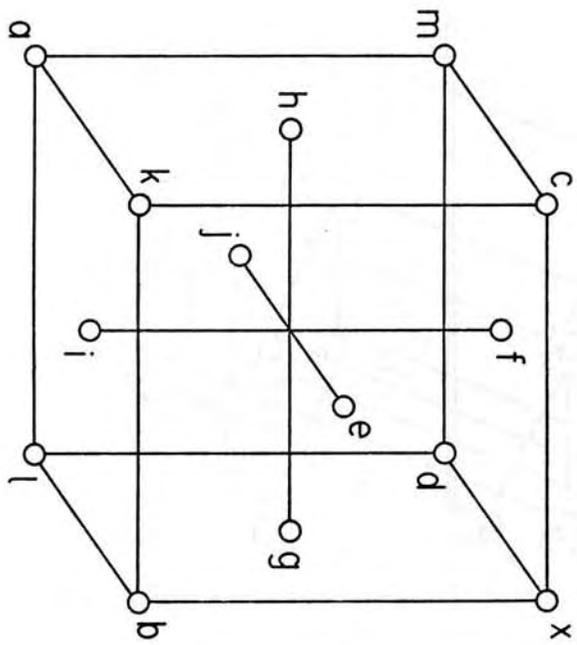


Figure 2

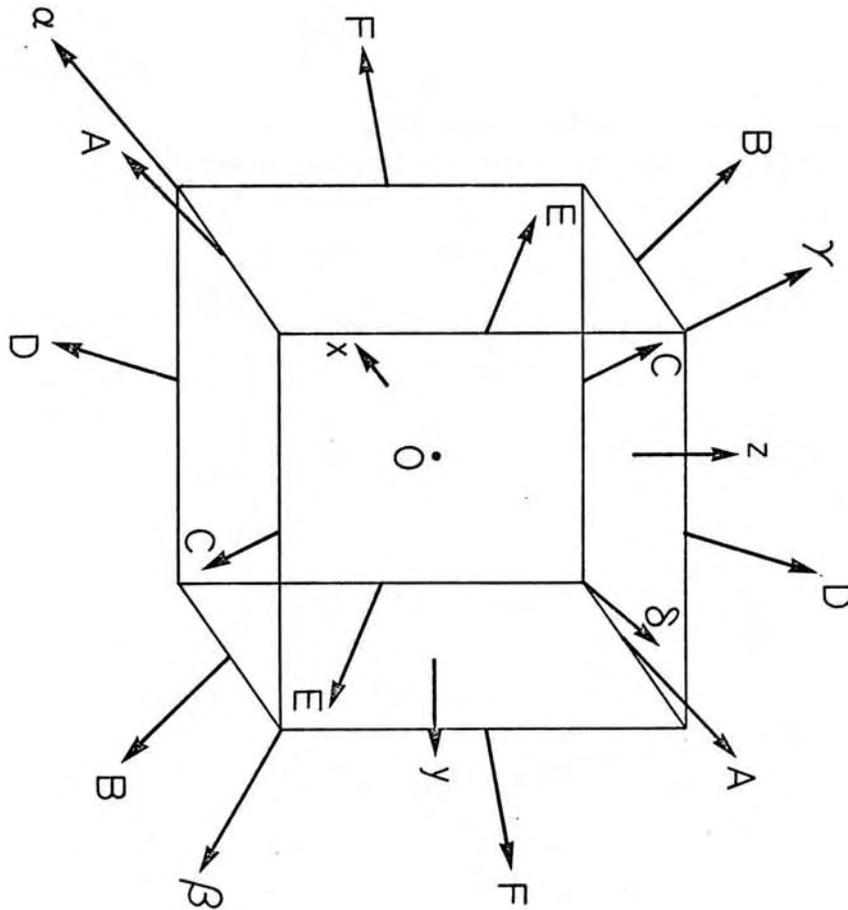


Figure 3

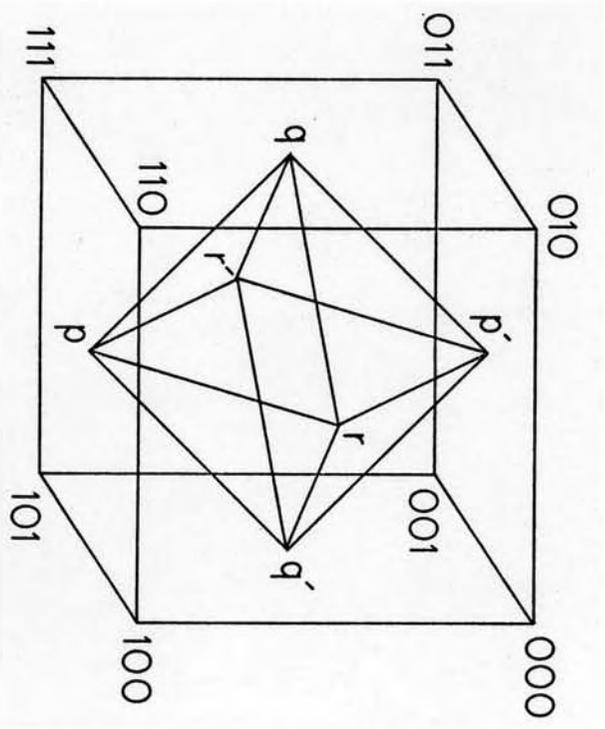


Figure 4