JOURNAL OF MATHEMATICAL PSYCHOLOGY 28, 282–316 (1984)

The Case of Piaget's Group INRC

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What is it that distinguishes Piaget's transformations N, R, and C from the rest of the 16! transformations of the 16 binary propositional operations? Here Piaget's INRC is considered as a subgroup of the group $\mathscr{M}_2$ of all automorphisms and dual automorphisms of the free Boolean algebra with two generators. This group is isomorphic to $S_4 \times C_2$. Its elements are given explicitly. Many other psychologically relevant subgroups of $\mathscr{M}_2$ play an important role. They are discussed and their connections shown. Particular attention is given to involutions, even if the view that they constitute the sole representation of reversibility is abandoned. Piaget's transformation R turns out not to be the inverse operation of relations. The group of automorphisms, dual automorphisms, anti-automorphisms of the algebra of binary relations on a finite set is found. A crystallographic presentation of these groups is given and related work by Bart (*Journal of Mathematical Psychology*, 1971, **5**, 539–553), Leresche (*Revue Européenne des Sciences Sociales*, 1976, **14**, 219–241), and Pólya (*The Journal of Symbolic Logic*, 1940, **5**, 98–103) is discussed.

ALGEBRAIC MODELS

My purpose in this paper is to dispel some misconceptions about Piaget's group INRC and thereby to open the way for a fruitful use of groups of so-called logical transformations and to model other domains of cognitive activities. The group INRC appears for the first time in Piaget (1950). Many comments and discussions have appeared since; but much of it is subtly vague, at best. If one wants to be more precise, it behooves one both to make clear what the group is meant to model and to become aware of its mathematical environment. This is not to say that the psychological and epistemological problems Piaget wanted to tackle are in fact mathematical problems. But it is useless, and may indeed be dangerous, to ask of psychology what mathematics by itself can accomplish. On the other hand, it is clear that one cannot reduce psychology to mathematics. The psychologist has to select the adequate mathematical structure for his problem. (This, of course, includes the possibility of deciding not to attempt any mathematical formalization.) But then he must be sufficiently aware of the mathematical possibilities which can be related to his subject matter. Sparse mathematics may blind one to interesting aspects of behaviour, adequate mathematics is full of suggestions for experiment. And that is another reason to discuss some of the various mathematical questions raised by the

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0022-2496/84 $3.00

investigation of the role played by the group INRC in the genetic psychology of logical operations. "Our main goal is [to contribute to] the construction of an adequate mathematical model, wherein one can make deductions and inferences strictly from the mathematical structure itself, and then emerge from the mathematical structure with interpretation hypothesis..." (Hoffman, 1980, p. 422).

The mathematical structures we want to discuss belong to algebra (and to its most elementary parts at that). Again this does not mean that other mathematical structures may not be useful for formalizing problems that arise in genetic psychology. But Piaget has favoured what he calls "logical formalization": "...logical formalization is absolutely essential every time we can carry out some formalization; ..." (Piaget, 1970, p. 10).

There are, however, at least two styles of logical formalization: the algebraic one and that of the theory of deduction. As far as Piaget is concerned, it seems correct to say that "... the author, joining the tradition of algebraic logic of Boole, Peirce, and Schröder, takes an algebraic view of the subject rather than the one of the theory of deduction" (Beth, 1950, p. 259). Within the context of Beth's virulent attack against the "Traité de Logique" (Piaget, 1949), this evocation, of what to Beth then appeared to be the rear guard of a defeated tradition, was clearly meant to deprecate Piaget's book. Nevertheless, it remains true that "logical formalization" for Piaget does mean, in the first place, algebraic formalization. Indeed, he says (Piaget, 1967, p. 269), "In view of the present lack of numerical units in psychology, the usefulness of logical models consists in providing a possible image of the very *operations* of the subject, as well as of the operational structures ("groupings," groups, lattices, etc.) which these operations form between themselves."

What he has in mind then, are qualitative models and not quantitative ones; not the mathematical disciplines related to psychometrics, but algebra.

In this paper, we shall deal with some algebraic structures of logical operations. Such structures, however, are not related exclusively to logic. Hence they may be used to characterize patterns of propositional as well as of experimental activity. One is thus led to the view that they characterize an "*algebra of competence*" detectable in the patterns of actions of a person reasoning or solving problems, doing things with propositions or with material objects.

For Piaget, the important algebraic structure that emerges during preadolescence and that explains most of the achieved and achievable performances is the INRC group.

To understand how this group arises, and what it stands for, we have to consider two among the acquisitions of preadolescence: the possibility of the child to view a given situation as one among several possible ones and his capacity to move from one of these possibilities to others related to the situation at hand.

The constitution of the set of possibilities that could be taken into account is a first step. As far as propositions are concerned, Piaget limits the *combinatorial system*—as he calls it—to the combinations of propositions present in a free Boolean algebra (mostly that generated by two propositions). It should be realized that this algebraic structure is the result of a great number of constraints (axioms) imposed on the interpropositional operations "and" or "meet," "or" or "join," and "negation" or "complement" (as will be seen in the next section). These constraints reduce the number of possibilities to a tractable amount and thus render the constitution of the combinatorial system both possible and useful. The question then arises of when this reduction is achieved and in which order the several constraints are introduced by the child (in other words, which are the intermediate algebraic structures). This however is not the subject matter of the present paper. Our problem is at *a level above that of the free Boolean algebra*, which we take for granted in this context. Indeed, Piaget not only posits the existence of the combinatorial system of all possibilities that *could* be taken into account by the child, but he examines also the means she has to *move* among these possibilities and *effectively* to take into account only those that are appropriate in a given situation. These, in most cases, do not make up the whole combinatorial system.

The *mobility* necessary for this is achieved by *operations on* the combinatorial manifold, that transform its elements among themselves. At the stage of formal operations, the transformations should be reversible and organized into a structured whole by associative composition. If we add that each transformation operates on any element of the manifold, then the structured whole is a *group*.

The notion of mobility will be made more explicit in the section entitled "Mobility, Orbits, Subgroups." One should also ask oneself in what order and under which circumstances the highly integrated structure of a group is built up from weaker (and more complicated) structures, and also whether any of these remain pertinent in some situations even at the stage of formal operations. These important questions are outside the scope of the present paper. Here the group structure of the transformations is taken for granted. The concept of reversibility, however, needs some elucidation.

Reversibility has been modeled by the fact that every element g of a group has an inverse g^{-1} such that $gg^{-1}=g^{-1}g=e$, where e is the unit element of the group (representing, in our case, the identity transformation, that leaves everything unchanged). The property of having an inverse has sometimes been interpreted erroneously, by Piaget and others. It is therefore necessary to discuss this point. Consider the operation J that transforms an element g of a group into its inverse g^{-1}, i.e., $Jg=g^{-1}$. Let us call J the "inversor." The inversor J *acts on* the elements of a given group (that may already, as in our case, be a group of transformations). It is not an element of *that* group. The inversor has the special property that, if it is applied twice to an element g, the result is again the same element g. This may be noted $J^2g=g$ and expresses simply the fact that the inverse of the inverse of an element g is again that element g, i.e., $(g^{-1})^{-1}=g$. In other words, the square J^2 of the inversor J is the identity transformation *on* the group in question (*not* belonging to that group). Transformations whose square, like that of the inversor, is equal to the identity, are called *involutions*. But on no account does the fact that the inversor is an involution mean that the reversibility of an operation g (i.e., the existence of an inverse operation g^{-1}) implies that it is an involution (i.e., that $g^2=e$).

The idea that every reversible operation is an involution appears at many places in

Piaget's writings. In particular Piaget's group INRC is a group that, in addition to the identity element, contains only involutions, and may be the reason why Piaget has considered INRC to be *the* group of logical transformations. This becomes particularly inconvenient, when one supposes that all changes that mark the passage from the stage of concrete operations to that of formal operations derive from that group considered as representing the "mental structure" or "algebra of competence" of the preadolescent. Thus, in "The Early Growth of Logic" (Inhelder and Piaget, 1958), the wealth contained in the psychological experiments has been trimmed to the size of INRC, much to the expense of the basic idea of an algebra of competence which this book was meant to promote.

Our first task, then, is to show that INRC is by no means the only group of logical transformations that it is meaningful to consider.

The Group INRC and the Automorphisms of FB(2)

Piaget (1972, p. XIV) "became aware of the existence of this group [INRC] within the propositional operations before 1949 (believing, moreover, that they were known), that is at a time when logicians still had not taken any interest in it."

This group, INRC, operates on the sixteen binary propositional operations (BPO) shown in the second column of Table 1. The operation N is the usual, Boolean, logical negation. "A second operation [R] may be called "reciprocity." We may ... call reciprocity the transformation that consists in negating the propositions which compose a given binary operation.... But reciprocity may be defined differently and I called it reciprocity with this second possible meaning in mind. In the case of the implication $p \to q$, the reciprocal is $q \to p$. Thus, the reciprocal of an implication results from permuting the two propositions occurring in the expression at stake" (Piaget, 1950, pp. 144, 145). The first definition is unambiguous, but the second is equivocal. Consider, e.g., the binary operation $p \vee q$. It is equivalent to $p' \to q$. Which are "the two propositions occurring in the expression"? If we consider them to be p' and q, then we get $q \to p'$ which is equivalent to $p' \vee q'$. This coincides with the result obtained by applying the first definition. This, then, is the meaning intended by Piaget. It is, however, not less natural to consider p and q as propositions occurring in $p \vee q$ or $q \to p'$, and this then remains unchanged when the second definition is used. The result is an operation, different from Piaget's R, and for which we shall introduce the sign T. Let us note that both R and T change $p \to q$ into $q \to p$. In this paper, the operation T will play a role at least as important as that of R. The operation C, finally, is defined as the product of N and R: $C = NR = RN$.

But in fact there are 16! (roughly $2 \cdot 10^{13}$) 1–1 transformations or permutations of the sixteen BPO, and not only four of them, since a priori the whole symmetric group S_{16}, the group of all permutations of 16 items has to be taken into account. The restriction to 1–1 transformations guarantees "reversibility," since for every such transformation t there exists an inverse t^{-1}.

However, most of these permutations are not interesting. It seems natural to consider the 16 BPO not simply as any 16 items, but to take into account how these BPO are related to each other. If one looks at the manner in which the 16 BPO can be obtained from two of them by using conjunctions, disjunctions, and negations, then one is led to consider the BPO as elements of a Boolean algebra, and more precisely, of a free Boolean algebra FB(2) with two generators. Consequently, one is then led to take into account only those transformations that "respect" this structure, in a sense which will be made clear presently.

Bart (1971) has imposed quite different conditions on the transformations of the elements of FB(2); the Boolean structure is not respected (cf. Appendix 3).

Let us recall here what a Boolean algebra is and what is meant by free. We do this in a way which permits easy comparison with operations on the set of relations we shall define below. Furthermore, making the algebraic ingredients of Boolean algebras explicit may suggest experiments concerning these features and about the gradual building-up of Boolean algebras from them.

A Boolean algebra

$$\mathscr{B} = \langle B, \wedge, \vee, ', 0, 1 \rangle$$

is a set B with two binary operations $\wedge$ (meet) and $\vee$ (join), one unary operation $'$ (complement) and two nullary operations (i.e., special elements) 0 and 1. These operations have properties stated and explained below.

(i) First consider $\mathscr{B}_\wedge = \langle B, \wedge, 1, 0 \rangle$.

(α) The operation $\wedge$ is associative, i.e., for all elements x, y, z of B,

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z. \tag{1}$$

(β) It has a unit element 1; this means

$$x \wedge 1 = x = 1 \wedge x \tag{2}$$

for all x.

Properties (α) and (β) are usually summarized by saying that $\mathscr{B}_\wedge$ is a monoid. However, this monoid has further properties.

(γ) It has a null element 0:

$$x \wedge 0 = 0 = 0 \wedge x. \tag{3}$$

(δ) It is commutative:

$$x \wedge y = y \wedge x. \tag{4}$$

(ε) It is idempotent:

$$x \wedge x = x \tag{5}$$

for any element x of B.

(ii) $\mathscr{B}_\vee = \langle B, \vee, 0, 1\rangle$ is a monoid (unit element 0) with null element 1, that is furthermore commutative and idempotent.

(iii) The operation $'$ is characterized by

(α) $(x')' = x,$ (6)

(β) $x \wedge x' = 0,\ x \vee x' = 1.$ (7)

(iv) The operations $\wedge$ and $\vee$ are linked in many ways.

(α) Absorption:

$$x \wedge (x \vee y) = x = x \vee (x \wedge y), \tag{8}$$

(β) Left distributivity (right distributivity then follows from commutativity):

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{9}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \tag{10}$$

(γ) De Morgan equalities

$$(x \wedge y)' = x' \vee y', \tag{11}$$

$$(x \vee y)' = x' \wedge y', \tag{12}$$

$$1' = 0, \tag{13}$$

and

$$0' = 1, \tag{13'}$$

as a consequence of (6).

We may also say that, (11), (12), (13), and (6) express the fact that $'$ is an isomorphism of the monoids $\mathscr{B}_\wedge$ and $\mathscr{B}_\vee$.

A Boolean algebra is free on n generators if it contains exactly all different elements that can be formed from the n generators by performing the operations $\wedge$, $\vee$, and $'$.

If we then consider the 16 BPO as elements of the free Boolean algebra FB(2) with two generators, it is reasonable to take into account not simply transformations but only transformations that conserve the Boolean structure, that is *Boolean automorphisms*. Such an automorphism ϕ is a one-to-one mapping of the set B onto itself such that

$$\phi(x \wedge y) = \phi x \wedge \phi y, \tag{14}$$

$$\phi(x \vee y) = \phi x \vee \phi y, \tag{15}$$

$$\phi(x') = (\phi x)', \tag{16}$$

$$\phi 0 = 0, \qquad \phi 1 = 1. \tag{17}$$

This limitation to automorphisms turns out to be altogether too drastic, since the transformations N and C, as defined and used by Piaget, are not automorphisms; they do not, for instance, conserve the special elements 0 and 1, as would be required by (17). Indeed Piaget defines N as being exactly the operation $'$. Hence, comparison of (13), (13′), and (17) shows that N is not an automorphism; and neither is C, since $C = RN = NR$. It will be easy to get hold of these and similar elements once the automorphisms have been found.

It is not difficult to find the automorphisms of a Boolean algebra. One has to take into account the fact that every finite Boolean algebra is isomorphic to the Boolean algebra of all subsets of a finite set; the operations are the union and the intersection of sets, the set theoretic complement, the void set, and the original set (e.g., Kurosh 1963, p. 189). If the set has n elements, the corresponding Boolean algebra will be noted $B(n)$. It has 2^n elements. A free Boolean algebra with n generators FB(n) has 2^p (with $p = 2^n$) elements and is isomorphic to $B(2^n)$. In the case of the 16 BPO we have $n = 2$: $\mathrm{FB}(2) \simeq B(4)$.

The set of all automorphisms of a Boolean algebra is a group under function composition. (A group is a monoid in which all elements have an inverse.) The group of all automorphisms of a finite Boolean algebra $B(n)$ can be obtained from, and is isomorphic to, the symmetric group S_n, the group of all permutations of the n elements of the corresponding set. More precisely, it is a subgroup of the group S_p ($p = 2^n$) of all permutations of the 2^n elements of $B(n)$. In particular, the automorphism group $\mathscr{A}_2$ of FB(2) is a subgroup of S_{16} isomorphic to S_4 and has 24 elements. For the subsequent discussions, we need to know how these transformations operate on the 16 elements of FB(2). It is also worthwhile to understand how to obtain them. We therefore introduce a short notation for each of the elements of FB(2). We derive it from the notation for the binary propositional operations introduced by Łukasiewicz (Bocheński, 1959, p. 11). He writes, for instance, *Apq* for $p \vee q$ and *Kpq* for $p \wedge q$. We denote the corresponding elements of FB(2) by a and k, respectively, and thus obtain the notation contained in Table 1. Furthermore, to each element we associate (column 3) a subset of a set of four elements, whose elements are designated by the same letters as the four elements k, l, m, and x. This establishes an isomorphism between FB(2) and the subsets of the set $\{k, l, m, x\}$ in which there corresponds to each subset the element of $BL(2)$ that is the disjunction of the elements of the subset. Thus, for instance, $\{k, l, m\}$ corresponds to $k \vee l \vee m = a$.

We now introduce the diagram of a finite Boolean algebra (often called its lattice). We say that an element x (different from y) *covers* y if

(i) $x \vee y = x$ and

(ii) there is no other element z such that

$$x \vee z = x, \qquad z \vee y = z.$$

Now connecting every element by an ascending line exactly to all elements that cover it, we obtain a diagram of a finite Boolean algebra. Figure 1 shows the lattice of

TABLE 1

Notations and Generators for the Free Boolean Algebra FB(2)

	(i, h)		(i, e)	(e, h)
v	1	$\{k, l, m, x\}$	1	1
a	$p \vee q$	$\{k, l, m\}$	$i \vee e'$	$e' \vee h$
b	$p \vee q'$	$\{k, l, x\}$	$i \vee e$	$e \vee h'$
c	$p' \vee q$	$\{k, m, x\}$	$i' \vee e$	$e \vee h$
d	$p' \vee q'$	$\{l, m, x\}$	$i' \vee e'$	$e' \vee h'$
i	p	$\{k, l\}$	i	$e \leftrightarrow h$
h	q	$\{k, m\}$	$i \leftrightarrow e$	h
e	$p \leftrightarrow q^a$	$\{k, x\}$	e	e
j	pwq^b	$\{l, m\}$	e'	e'
g	q'	$\{l, x\}$	iwe	h'
f	p'	$\{m, x\}$	i'	ewh
k	$p \wedge q$	$\{k\}$	$i \wedge e$	$e \wedge h$
l	$p \wedge q'$	$\{l\}$	$i \wedge e'$	$e' \wedge h'$
m	$p' \wedge q$	$\{m\}$	$i' \wedge e'$	$e' \wedge h$
x	$p' \wedge q'$	$\{x\}$	$i' \wedge e$	$e \wedge h'$
o	0		0	0

Note. For columns 4 and 5 see under the heading "Relations."

[a] $p \leftrightarrow q = (p \wedge q) \vee (p' \wedge q')$.

[b] $pwq = (p \wedge q') \vee (p' \wedge q)$.

FB(2) thus obtained. Given such a lattice, joins and meets of any two elements can easily be found. The join (meet) of two elements is the element at which the ascending (descending) lines starting from these elements meet.

To obtain from the group S_4 of permutations of the four elements k, l, m and x, the isomorphic subgroup $\mathscr{A}_2$ of S_{16}, one simply has to find how the elements of S_4 transform the subsets of $\{k, l, m, x\}$. Take, for instance, the transformation (kx) (lm) that exchanges first l and m, and then k and x. This transformation induces, for

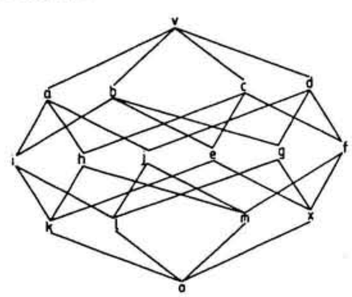

FIG. 1. Lattice of FB(2).

example, a transformation of the set $\{k, l\}$ into the set $\{m, x\}$, that is (according to Table 1), of the element i into the element f; the set $\{k, l, x\}$ is transformed into $\{k, m, x\}$, that is, b is transformed into c—and so on. We thus find that $(kx)(lm) \in S_4$ induces

$$(kx)(lm)(ad)(bc)(if)(hg)(e)(j)(v)(o) \in S_{16}.$$

It may be seen that this is Piaget's transformation R, e.g., in (Piaget, 1972, p. 254).

This way, we obtain the whole group $\mathscr{A}_2$ of automorphisms of FB(2) shown in Table 2. The elements of $\mathscr{A}_2$ are represented as products of cycles. The cycles are applied to the elements of FB(2) from right to left. A cycle $(lkxm)$, for instance, transforms l into k, k into x, x into m, and m into l: $(lkxm) = (lk)(kx)(xm)$.

A permutation X is said to have order n, if performed n times it gives the identity permutation I $(X^n = I)$. The group $\mathscr{A}_2$ contains elements of orders two, three, and four. The 24 elements are divided into five groups; these are conjugate classes of S_4 defined and used below. We have not indicated how the automorphisms transform the

TABLE 2

Group $\mathscr{A}_2$ of Automorphisms of FB(2)

S_4	Extended to S_{16}		Generators
$(k)(l)(m)(x)$	$(a)(b)(c)(d)$	$(i)(h)(e)(j)(g)(f)$	I
$(mx)(k)(l)$	$(ab)(c)(d)$	$(he)(jg)(i)(f)$	$UTU = VP$
$(lx)(k)(m)$	$(ac)(b)(d)$	$(ie)(jf)(h)(g)$	U
$(kx)(l)(m)$	$(ad)(b)(c)$	$(ig)(hf)(e)(j)$	$VUV = TR$
$(lm)(k)(x)$	$(bc)(a)(d)$	$(ih)(gf)(e)(j)$	T
$(km)(l)(x)$	$(bd)(a)(c)$	$(ij)(ef)(h)(g)$	$TVT = UQ$
$(kl)(m)(x)$	$(cd)(a)(b)$	$(hj)(eg)(i)(f)$	V
$(kl)(mx)$	$(ab)(cd)$	$(hg)(ej)(i)(f)$	$UTUV = P$
$(km)(lx)$	$(ac)(bd)$	$(if)(ej)(h)(g)$	$VUVT = Q$
$(kx)(lm)$	$(ad)(bc)$	$(if)(hg)(e)(j)$	$TVTU = R$
$(lxm)(k)$	$(abc)(d)$	$(ieh)(jgf)$	TU
$(kxm)(l)$	$(abd)(c)$	$(igj)(hef)$	$TUVT$
$(lmx)(k)$	$(acb)(d)$	$(ihe)(jgf)$	UT
$(lkx)(m)$	$(acd)(b)$	$(ieg)(jhf)$	UV
$(kmx)(l)$	$(adb)(c)$	$(ijg)(ehf)$	$TVUT$
$(lkx)(m)$	$(adc)(b)$	$(ige)(jfh)$	VU
$(lkm)(x)$	$(bcd)(a)$	$(ihj)(efg)$	TV
$(lmk)(x)$	$(bdc)(a)$	$(ijh)(egf)$	VT
$(lkxm)$	$(abcd)$	$(iefj)(hg)$	$TUV = UP = RU$
$(lxmk)$	$(abdc)$	$(igfh)(ej)$	$VTU = TQ = PT$
$(lmkx)$	$(acbd)$	$(jheg)(if)$	$UVT = VR = QV$
$(lkmx)$	$(acdb)$	$(ihfg)(ej)$	$UTV = QT = TP$
$(lxkm)$	$(adbc)$	$(jgeh)(if)$	$TVU = RV = VQ$
$(lmxk)$	$(adcb)$	$(ijfe)(hg)$	$VUT = PU = UR$

elements v and o. According to (17) they remain unchanged. Thus, (v) (o) should be added to each automorphism.

Some of these automorphisms have been given special symbols (P, Q, R, U, T, V). They are *involutions* (there are three other involutions in $\mathscr{A}_2$). Involutions are transformations X of order two: $X^2 = I$.

Such transformations are equal to their inverse: $X = X^{-1}$. Piaget calls them also "reciprocities."

The important role played by involutions in the formalization of Piaget's psychology and epistemology has been stressed by several authors (see, e.g., Wermus, 1972). But the groups of transformations considered were always *groups consisting of involutions* (i.e., Boolean groups). The group $\mathscr{A}_2$ contains 13 such groups and $\mathscr{M}_2$ relevant to the discussion of the transformations of the BPO (cf., subsequent section) contains 48 of them, of which 25 are isomorphic to INRC. Thus, INRC has no special status a priori as a group of logical transformations.

It has often been considered, that inversible operations are represented by involutions. However, "this 'strong' view of reversibility is derived not from extensive empirical examinations of reversibility behaviors but rather from a consideration of the reversibility indicated in the INRC group" (Bart, 1971, p. 542). As discussed in the first section, this view probably arises from a misinterpretation of the fact that the "inversor" is an involution.

It is clear that transformations of orders higher than two play a role in the functioning of intelligence. In such cases, reversibility simply corresponds to the existence of an inverse transformation, that however need not be equal to the original one.

Nevertheless, giving up the strong view of reversibility, the exclusive privilege of involutions to represent reversibility, does not mean abandoning involutions altogether. They still play an important role in the groups underlying a person's intellectual (and other) activities. The group $\mathscr{A}_2$ of automorphisms of FB(2) is a good illustration of what may happen. This group is not a group of involutions, but a *group generated by involutions*, the three involutions U, T, and V for instance. That is, every element of $\mathscr{A}_2$ *can* be written as a product of these three elements (generators), as shown in the column "generators" of Table 2. It can be seen there that no element is the product of more than four factors.

A more complicated structure than that of Boolean groups arises because *the generating involutions do not commute*. Noncommutativity and the appearance of higher order (i.e., noninvolutive) operations are probably closely related genetically. Now, noncommutativity is certainly a basic experience in the growth of intelligence. Below, in the discussion of subgroups, we shall find a typical example of noncommutativity.

Involutive generators of $\mathscr{A}_2$ other than U, T, and V may be selected. Some of these selections will be discussed below. Depending on the choice of generators, some automorphisms appear more complicated than others, since they are represented as products of many factors. Such a notational complication may reveal psychological complexity. One may even require that it be so; that is, that the generators be selected in such a way that the genetically simple appears as simply generated. There is, of course, also the possibility of choosing as generators elements other than involutions. The possibilities are numerous, and to point this out is all we can do here. The algebraic model must be adapted to actual experience.

We conclude this section by remarking that, quite generally, $\mathscr{A}_n$, the group of automorphisms of the free Boolean algebra FB(n) with n generators is isomorphic to S_p ($p = 2n$) and that it is generated by $2^n - 1$ involutions. It is also useful to know that it may be viewed as a group of *crystallographic transformations* of a $(2^n - 1)$-dimensional crystal. In the case we are mainly interested in ($n = 2$), the crystal is an ordinary 3-dimensional cubic crystal. More about the crystallographic representation may be found in Appendix 1.

Dual Automorphisms

Piaget's transformations N and C are not automorphisms of FB(2), but *dual automorphisms* (d-automorphisms for short), that is, transformations that fulfil the following conditions:

$$\Psi(x \wedge y) = \Psi x \vee \Psi y, \tag{18}$$

$$\Psi(x \vee y) = \Psi x \wedge \Psi y, \tag{19}$$

$$\Psi(x') = (\Psi x)' \tag{20}$$

$$\Psi 0 = 1, \qquad \Psi 1 = 0. \tag{21}$$

We may now say that Eqs. (11), (12), and (13) express the fact that $'$ is a d-automorphism. Furthermore, we note that Piaget's transformation N, the Boolean negation, is the operation $'$:

$$Nx = x'; \tag{22}$$

thus, it is indeed a d-automorphism. It has the further property of commuting with all automorphisms; equations (16) and (22) give together

$$\phi N(x) = N\phi(x). \tag{23}$$

It is important to note that the product of an automorphism and a d-automorphism is a d-automorphism. Piaget's C, which is equal to RN, is an example. Furthermore, as it is easy to convince oneself, the product of two d-automorphisms is an automorphism. (The d-automorphisms do not form a group.) It follows that in a group containing automorphisms and d-automorphisms, there is the same number of both of them; the order of such a group is necessarily even. We shall call such a group, *a* group of "mixed automorphisms". As we shall see, the d-automorphisms of such a group are not necessarily those obtained through multiplication by N of its automorphisms. In other words, a group of mixed automorphisms is not always a "direct product" of its subgroup of automorphisms by the group $\{I, N\}$.

If among the d-automorphisms of a group of mixed automorphisms, there is one involution, say X, that commutes with all automorphisms, then the group is a direct product of its subgroup of automorphisms with the group of order two generated by that element. Equation (23) shows N is such an element. Therefore, the group $\mathscr{M}_2$ of all mixed automorphisms of FB(2) is the direct product of $\mathscr{A}_2$ and the group $\langle N \rangle$ generated by N; it is isomorphic to $S_4 \times C_2$:

$$\mathscr{M}_2 = \mathscr{A}_2 \times \langle N \rangle \simeq S_4 \times C_2. \tag{24}$$

In other words, each d-automorphism is the product of an automorphism with N. (But in a subgroup of $\mathscr{M}_2$ these automorphisms may be absent.) The action of N is known from (22); thus, all d-automorphisms can be computed by multiplication of permutations. The result is shown in Table 3.

Since $\mathscr{A}_2$ was generated by three involutions, it follows from the construction, that

TABLE 3

Dual Automorphisms of FB(2)

Elements of S_{16}		Generators
$(ax)(bm)(cl)(dk)$	$(if)(hg)(ej)$	N
$(am)(bx)(cl)(dk)$	$(if)(hj)(eg)$	$V\bar{P}$
$(al)(bm)(cx)(dk)$	$(ij)(hg)(ef)$	$\bar{U} = UN$
$(ak)(bm)(cl)(dx)$	$(ih)(fg)(ej)$	$T\bar{R}$
$(ax)(bl)(cm)(dk)$	$(ig)(hf)(ej)$	$\bar{T} = TN$
$(ax)(bk)(cl)(dm)$	$(ie)(jf)(hg)$	$U\bar{Q}$
$(ax)(bm)(ck)(dl)$	$(if)(he)(jg)$	$\bar{V} = VN$
$(am)(bx)(ck)(dl)$	$(if)(h)(e)(j)(g)$	$\bar{P} = PN$
$(al)(bx)(ck)(dm)$	$(hg)(i)(e)(j)(f)$	$\bar{Q} = QN$
$(ak)(bl)(cm)(dx)$	$(ej)(i)(h)(g)(f)$	$\bar{R} = RN = C$
$(amcxbl)(dk)$	$(ijhfeg)$	TUN
$(amdxbk)(cl)$	$(ihjfge)$	$TUVTN$
$(albxcm)(dk)$	$(igefhj)$	UTN
$(aldxck)(bm)$	$(ijgfeh)$	UVN
$(akbxdm)(cl)$	$(iegfjh)$	$TVUTN$
$(akcdkl)(bm)$	$(ihefgj)$	VUN
$(bldmck)(ax)$	$(igjfhe)$	TVN
$(bkcmdl)(ax)$	$(iehfjg)$	VTN
$(amck)(bldx)$	$(ijfe)(h)(g)$	$TUVN = \bar{R}U$
$(amdl)(bkcx)$	$(ihfg)(e)(j)$	$VTUN = \bar{P}T$
$(albk)(cmdx)$	$(ehjg)(i)(f)$	$UVTN = \bar{Q}V$
$(aldm)(bxck)$	$(igfh)(e)(j)$	$UTVN = T\bar{P}$
$(akbl)(cxdm)$	$(egjh)(i)(f)$	$TVUN = V\bar{Q}$
$(akcm)(bxdl)$	$(iefj)(h)(g)$	$VUTN = \bar{U}R$

$\mathcal{M}_2$ is generated by four of them, U, T, V, and N, for example. However, we shall see that it can be generated by three involutions only, and in a given context this will appear as a natural way of generation.

The 24 d-automorphisms form five conjugate classes of $S_4 \times C_2$. Note that there is a class of elements of order six. Again, we have not indicated the action of the d-automorphisms on v and on o. According to (21), v and o are exchanged in all cases. Thus, (vo) should be added to each d-automorphism.

Generally, the group $\mathcal{M}_n$ of all mixed automorphisms of FB(n) is given by

$$\mathcal{M}_n = \mathcal{A}_n \times \langle N \rangle \simeq S_p \times C_2 \tag{25}$$

(with $p = 2^n$). It can be generated by $2^n - 1$ involutions.

We have thus found a group $\mathcal{M}_2$ of 48 elements operating on FB(2). For various reasons it may be tempting to want to deal with smaller and simpler groups. However, reasons for doing so have to be advanced, at least tentatively. The restriction must not simply be obtained through unawareness of other possibilities or by decree. In the following two sections, we seek for criteria permitting to restrict the group $\mathcal{M}_2$ to some of its subgroups.

Mobility, Orbits, Subgroups

In this section we discuss various subgroups of the group of mixed automorphisms $\mathcal{M}_2$ and introduce a convenient way of describing how they operate on FB(2).

First we want to illustrate the role played by such groups in experimental situations. It will be best to quote from "The Psychology of the Child" (Inhelder and Piaget, 1969, p. 139), where we find a convenient illustration. "Let us take as an example the implication $p \to q$, and let us imagine ... [that] a child between 12 and 15 ... observes a moving object that keeps starting and stopping.... He notices that the stops seem to be accompanied by the lighting of an electric bulb. The first hypothesis he will make is ... $p \to q$ (light implies stop). There is only one way to confirm the hypothesis, and this is to find out whether the bulb ever lights up without the object stopping or $p \wedge q'$ [$=N(p \to q)$].... But he may also wonder whether the light instead of causing the stop is caused by it or $q \to p$ (now the reciprocal [$R(p \to q)$] of $p \to q$). To confirm $q \to p$ (stop implies light), he looks for the opposite case which would disconfirm it; that is, does the object ever stop without the light going on? This case, $p' \wedge q$, is the inverse [$N(p \to q)$] of $p \to q$. It is at the same time a correlative [$C(p \to q)$] of $p \to q$."

Inhelder and Piaget describe here a set of four elements of FB(2) that are transformed among themselves by the group INRC, i.e., what technically is called an *orbit* of that group on FB(2). This orbit, then, is the set $\{b, c, l, m\}$. The other orbits of INRC are $\{a, d, k, x\}$, $\{i, f\}$, $\{h, g\}$, $\{e, j\}$ and $\{o, v\}$. Orbits containing four elements are termed "quaternes" by Piaget (1950) and "squares of quaternality" by Gottschalk (1953).

In experimental situations one observes orbits, the subsets of those among the possibilities of the combinatorial manifold, that are effectively taken into account by a person in a given situation. From the orbits one may infer the group that generates them. The orbits are the expression of the mobility embodied in the group of transformations, acting on the combinatorial system. "Thus, without knowing any logical formula or the formal criteria for a mathematical "group"..., the pre-adolescent of 12 to 15 is capable of manipulating transformations according to the four possibilities I, N, R, and C." (Inhelder & Piaget, 1969, p. 140.)

The "quaterne" $\{b, c, l, m\}$ seems to accompany the use of the implication in experimental situations. It is therefore noteworthy that there are four other subgroups of $\mathscr{M}_2$ besides INRC (G_4) that have this same orbit, as may be seen in Table 4, where the orbits on FB(2) of all the subgroups of $\mathscr{M}_2$ discussed in this section may be found.

Note that even the largest group $\mathscr{M}_2$ has more than one orbit. Groups of permutations that have a single orbit are termed "transitive." It is not clear whether transitivity should be considered as an advantage or rather as a handicap. According to the interpretation of orbits proposed here, transitivity would mean that a person would actually take into account all possibilities of the combinatorial manifold. (For another view of transitivity, see Appendix 3.) Anyhow, the groups $\mathscr{M}_n$ and $\mathscr{A}_n$ are not transitive on FB(n), so that a structure-preserving group cannot be transitive, and a transitive group cannot be structure preserving.

Before discussing subgroups of $\mathscr{M}_2$ and their orbits, let us summarize the view of mobility of thought that has been advanced here. There are two aspects to it. (i) The mobility provided by the combinatorial system of all possibilities that could be taken into account: this system is the outcome not only of new freedom acquired but also of constraints that limit the possibilities to a tractable amount. (ii) Transformations that operate on the combinatorial manifold and that are not specific to elements of

TABLE 4

Orbits of Groups of Mixed Automorphisms

$\mathscr{M}_2$	$\{abcd \mid klmx\}\{ihejgf\}$
G_1	$\{abcd \mid klmx\}\{ihgf\}\{ej\}$
G_2	$\{abcd \mid klmx\}\{if\}\{hg\}\{ej\}$
G_3	$\{ad \mid kx\}\{bc \mid lm\}\{ihgf\}\{e \mid j\}$
G_p	$\{ab \mid kl\}\{cd \mid mx\}\{i\}\{f\}\{hg\}\{ej\}$
G_q	$\{ac \mid km\}\{bd \mid lx\}\{if\}\{ej\}\{h\}\{g\}$
G_4	$\{ad \mid kx\}\{bc \mid ml\}\{if\}\{hg\}\{e\}\{j\}$
G_5	$\{a \mid k\}\{bc \mid lm\}\{d \mid x\}\{ih\}\{fg\}\{e \mid j\}$
G_6	$\{ad \mid kx\}\{b \mid l\}\{c \mid m\}\{ig\}\{hf\}\{e \mid j\}$
G_7	$\{a \mid x\}\{d \mid k\}\{bc \mid lm\}\{ih \mid gf\}\{e \mid j\}$
G_8	$\{ad \mid kx\}\{bc \mid lm\}\{if \mid hg\}\{e \mid j\}$
G_9	$\{a \mid x\}\{b \mid m\}\{c \mid l\}\{d \mid k\}\{i \mid f\}\{h \mid g\}\{e\}\{j\}$

Note. Under the subgroups of automorphisms of these groups, some of the orbits split into two orbits. The splitting is indicated by a vertical line. The orbit $\{v \mid o\}$ has been omitted in all cases.

the manifold but apply equally to all of them and form a group: the group, however, is restricted by the requirement of preserving some or all of the structure that has been used in the construction of the combinatorial system. As a result, the group, although its transformations are defined everywhere on the system, does not necessarily provide the means of transforming an element of the system into any other element.

To distinguish subgroups of $\mathscr{M}_2$, we shall now examine how they act on the two generators of FB(2). There are several possibilities of choosing these generators, essentially three possibilities, since with each generator its complement also occurs. We have thus the three systems of generators,

$$\{p, q\} \quad \text{or} \quad \{i, h\}$$
$$\{p, p \leftrightarrow q\} \quad \text{or} \quad \{i, e\}$$
$$\{p \leftrightarrow q, q\} \quad \text{or} \quad \{e, h\}.$$

The group $\mathscr{A}_2$ of all automorphisms (and, a fortiori, the group $\mathscr{M}_2$) exchanges the systems of generators. The automorphisms U and V, for instance, transform the pair $\{i, h\}$ into $\{e, h\}$ and $\{i, e'\}$, respectively. It seems reasonable, therefore, to investigate those transformations that do not change one chosen system of generators into another. As for the choice of the system, it appears most natural to take $\{p, q\}$ (the "standard generator system"). We are thus selecting now those transformations contained in Tables 2 and 3 that permute the elements i, h, f, and g only among themselves (and, by the way, do the same to the remaining 12 elements of FB(2)). We find the 16 transformations detailed in the second and third column of Table 5.

TABLE 5

Automorphisms and d-Automorphisms that Preserve the Generator System $\{p, q\}$

Permutations	Generators		Crystallographic Interpretation	
$(p)(q)(p')(q')$	I	$\bar{R} = C$	1	m_x
$(pp')(q)(q')$	Q	$\bar{P}$	2_y	m_z
$(qq')(p)(p')$	$TQT = P$	$\bar{Q} = T\bar{P}T$	2_z	m_y
$(pp')(qq')$	$QTQT = R$[a]	N	2_x	$\bar{1}$
$(pq)(p'q')$	T	$T\bar{R}$	m_B	2_A
$(pq')(qp')$	$QTQ = \bar{P}T\bar{P}$[b]	TN	m_A	2_B
$(pqp'q')$	QT	$\bar{P}T$	$\bar{4}_x$	4_x^3
$(pq'p'q)$	TQ	$T\bar{P}$	$\bar{4}_x^3$	4_x

Note. The crystallographic interpretation is presented in Appendix 1.

[a] $QTQT = PTPT = R$.

[b] $QTQ = \bar{P}T\bar{P} = TR$.

The 16 elements form a subgroup G_1 of $\mathscr{M}_2$; they make up the invariance group of the generator system $\{p, q\}$. However, they act on the generators p and q only in eight distinguishable ways, as shown in the first column of Table 5. For each automorphism, there is a d-automorphism, that acts in the same way on the set $\{p, q, p', q'\}$, but differs, of course, in its action on the remaining elements of FB(2). The d-automorphism $C = \bar{R} = RN$, for instance, reduces to the identity transformation when restricted to the above set. Thus, any automorphism and the d-automorphism resulting from it by multiplication with $\bar{R}$ have the same effect on the generator system $\{p, q\}$. The action of G_1 is the same as that of its subgroup of automorphisms G_1' (second column of Table 5) or, indeed, as that of the three other subgroups of G_1 that are isomorphic to it; all are isomorphic to D_4. We shall refer to G_1 and G_1' as "restricted groups" of, respectively, mixed automorphisms and automorphisms of FB(2). The whole group G_1 is generated by multiplying the elements of D_4 by N (or by $\bar{R} = C$). The structure of G_1 is thus that of a direct product

$$G_1 \simeq D_4 \times C_2. \tag{26}$$

Recall that the dihedral group D_n $(n \geqslant 3)$ is a group of $2n$ elements that is isomorphic to the group of symmetries of a regular polygone of n sides. It is not commutative and is generated by two involutions x and y obeying the defining equations

$$x^2 = y^2 = (xy)^n = I. \tag{27}$$

For $n = 2$ the equations also define a group, the commutative group D_2. Piaget's INRC is isomorphic to D_2. Groups isomorphic to D_2 can be found in many situations and D_2 has been given various names, of which "Klein's four-group," or simply "four-group" are found most frequently.

The fact that x and y do not commute for $n \neq 2$ is shown most clearly by

$$xy = (yx)^{n-1} \tag{28}$$

an immediate consequence of (27). In columns two and three of Table 5, we have displayed the involutions Q and T (and also $\bar{P}$ and T) as generators of a subgroup isomorphic to D_4. Equation (27) here specializes to

$$TQ = (QT)^3 \tag{29}$$

and this shows how the noncommutativity comes about. The operations of exchanging the two generators p and q (by T) and that of exchanging p with its complement p' (by Q) (i.e., negating one of the generators) do not commute. The automorphism P that negates (complements) the other generator q, does not commute with T either. This may exemplify one of the most primitive experiences of noncommutativity: exchanging the two members of an ordered pair (this is an involution) and doing something to one member are operations that do not commute, and performing one after the other results in an operation that is not an involution, even though one starts with two involutions. It may be interesting to investigate early experiences of noncommutativity. The four subgroups of G_1 isomorphic to D_4 mentioned above are the groups generated by (for instance) Q and TN, Q and T, $\bar{P}$ and TN, and finally $\bar{P}$ and T. Let us use the last group to illustrate the fact already mentioned, namely, that a group of mixed automorphisms is not necessarily the direct product of its subgroup of automorphisms with the group $\langle N \rangle$ generated by the Boolean negation N. In our group, there are four automorphisms I, $T\bar{P}T\bar{P}$, T, and $\bar{P}T\bar{P}$; the d-automorphisms $\bar{P} = PN$, $T\bar{P}T$, $\bar{P}T$, and $T\bar{P}$ are not obtained by multiplying by N the automorphisms of the group (i.e., the transformations P, TPT, PT, and TP do not belong to the group in question).

It is plausible to think that the restricted groups G_1 and G_1' represent the mobility of thought that a person of the formal stage (i.e., a nonspecialized person) may attain. However, in the gradual building-up of competence, sub-groups certainly play an important role. We now proceed to discuss some of them.

We have already met with the group

$$G_9 = \{I, \bar{R}\} = \langle \bar{R} \rangle \tag{30}$$

generated by $\bar{R}$. It is the subgroup that leaves p and q as well as their complements unchanged.

Studying Table 5, we can characterize some other subgroups of G_1. The group that fixes the generator p (as well as p') and the one that fixes q and q' are, respectively,

$$G_p = \{I, P; \bar{R}, \bar{Q}\} = \langle P \rangle \times \langle \bar{R} \rangle \tag{31}$$

$$G_q = \{I, Q; \bar{R}, \bar{P}\} = \langle Q \rangle \times \langle \bar{R} \rangle. \tag{32}$$

The group G_p is generated by P and $\bar{R}$, G_q by Q and $\bar{R}$. The elements listed after the semicolon are d-automorphisms. The generators are exchanged (and their complements also) by

$$G_5 = \{I, T; \bar{R}, T\bar{R}\} = \langle T \rangle \times \langle \bar{R} \rangle. \tag{33}$$

The two generators are changed into their complements by

$$G_4 = \{I, R; \bar{R}, N\} = \langle R \rangle \times \langle \bar{R} \rangle. \tag{34}$$

This is Piaget's INRC. All these groups are isomorphic to Klein's four group D_2, but the last two deserve a special status because of their simple action on both generators. They also have in common Piaget's "quaterne," the orbit $\{b, c, l, m\}$. Two further groups, isomorphic to D_2, that have such an orbit are

$$G_7 = \{I, T; N, \bar{T}\} = \langle T \rangle \times \langle \bar{T} \rangle$$

and

$$G_8 = \{I, R; \bar{T}, R\bar{T}\} = \langle R \rangle \times \langle \bar{T} \rangle.$$

Note that the permutations R and T play similar roles in these four groups—for example, G_7 is the "T-analogue" of Piaget's INRC. In the last section on relations, we shall see that T not R, as Piaget thought, corresponds to the operation of conversion of a relation.

Generators are fixed or exchanged with their complements by

$$G_2 = G_p \vee G_q = \{I, P, Q, R; N, \bar{P}, \bar{Q}, \bar{R}\} = \langle P\rangle \times \langle Q\rangle \times \langle N\rangle. \tag{35}$$

Note that G_2 is not the set-theoretic union (this would not be a group), but the group generated by the groups in question. Finally, the group that exchanges generators and also their complements or changes generators into complements is

$$G_3 = G_4 \vee G_5 = \{I, R, T, TR; N, \bar{R}, TN, T\bar{R}\} = \langle R\rangle \times \langle T\rangle \times \langle N\rangle. \tag{36}$$

We shall meet this group when discussing structure preserving transformations of relations. It is also the largest group having Piaget's "quaterne" as orbit. Both the groups G_2 and G_3 are isomorphic to $D_2 \times C_2 = C_2 \times C_2 \times C_2$.

How these groups are related, is shown in the lattice of Fig. 2 of subgroups of the group of mixed automorphisms $\mathscr{M}_2$. It is a *sublattice* of the lattice of *all* subgroups of $\mathscr{M}_2$. The groups on the right are groups of mixed automorphisms, those on the left are automorphisms. The five groups having an orbit $\{b, c, l, m\}$ are boxed. Among them is Piaget's G_4. The lines linking a group to a supergroup (above) and a subgroup (below) may also suggest pathways for the gradual building-up of these transformation groups during the development of intelligence.

All the subgroups of $\mathscr{M}_2$ and of $\mathscr{A}_2$ discussed so far concern the generator system $\{i, h\}$. The corresponding subgroups for the systems $\{i, e\}$ and $\{e, h\}$ are obtained in the following way. The automorphism $U(VP)$ exchanges i and e (h and e); the

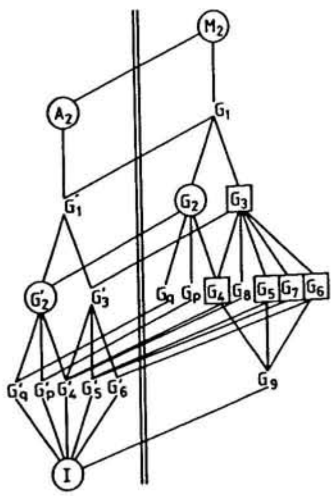

FIG. 2. Lattice of subgroups of $\mathscr{M}_2$.

subgroups fixing the generator system $\{e, h\}$ ($\{i, e\}$) are conjugates by U (VP) of the subgroups fixing $\{i, e\}$. Recall that the conjugate of a subgroup G of $\mathcal{M}_2$ by an element $x \in \mathcal{M}_2$ is the set $xGx^{-1} = \{xgx^{-1} : g \in G\}$. Another similar construction will be of interest. The set $C_g = \{xgx^{-1} : x \in G\}$ of all elements conjugate to a given element g, is termed the *conjugate class* of g. Conjugate classes are either disjoint or coincide. Tables 3, 4, and 5 display this partition into conjugate classes.

If x is not in G, then the group xGx^{-1} may be different from G or equal to G. It is equal when G is a normal (self-conjugate) subgroup, otherwise it is different. In the lattice of Fig. 2, the normal subgroups are placed in circles. These groups are common to the three generator systems. The only case not obvious from the beginning is that of the group $G_2 = \langle P \rangle \times \langle Q \rangle \times \langle N \rangle$, and of its subgroup of automorphisms G_2'. These groups fix the generators or exchange them with their conjugates in any generator system.

A word of caution is needed for the groups G_5 and its subgroup of automorphisms G_5' that exchange the generators and also their complements. They have six conjugates and not only three like the other non-normal subgroups. One should add to the subgroups already considered the subgroup G_6 that exchanges a generator with the complement of the other:

$$G_6 = \{I, TR; \bar{R}, \bar{T}\} = \langle TR \rangle \times \langle C \rangle.$$

The pair (G_5, G_6) is transformed by conjugation into the pairs corresponding to the other generator systems.

With the help of the crystallographic translations of Appendix 1, all these groups may be visualized as transformations of a cube. The change of generator system, particularly, becomes easy to grasp.

In addition to the gradual building-up of the restricted group G_1 and, ultimately, perhaps the (unrestricted) group $\mathcal{M}_2$ suggested by the lattice of Fig. 2, one may also wish to consider the successive construction of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots$, or the corresponding restricted groups. Bart (1971), for instance, has suggested that a person's capacity for combining propositions may be measured by the number n of generators of FB(n), or more adequately, by the number of generators of the group operating on FB(n). Bart's construction of that group will be discussed in Appendix 3. If we take the group $\mathcal{M}_n$, the number of generators increases as 2^n so that, when progressing from n to $n+1$, we have to add 2^n new generators. The same occurs in the case of Bart's group Φ_n, although the structure of his groups is quite different. In Appendix 2, we present Pólya's construction (1937, 1940) of the restricted group of automorphisms of FB(n). The situation improves radically. The number of generators is n; at each step, we have to add one more generator.

Relations and Their Structure Preserving Transformations

For Piaget the group INRC is the *synthesis* of two kinds of reversibility, and of the two domains where the two reversibilities N and R originate. The one, N, corresponds to the taking of complements in the domain of classes and, at the stage of formal operations, becomes Boolean complementation (or negation). The other domain is that of binary relations and the operation is that of inversion (or conversion) of binary relations. Piaget thought that R is the operation that brings about conversion.

In this section we shall argue that T and not R corresponds to conversion; we shall examine the "algebra of relations" as being a *synthesis* of the Boolean structure (of which N is characteristic) and of the "Peircean" structure of relations to which T pertains; and we shall determine the structure-preserving, reversible transformations of the algebra of relations.

First, however, let us illustrate some notions concerning binary relations. In connection with Fig. 1 we have already introduced a binary relation, the relation K "covers." We see in Fig. 1, e.g., that a covers h or, aKh; and only a covers h. The converse binary relation $\breve{K}$ is "is covered by." We see that h is covered by a, $h\breve{K}a$. It is clear that hKa is true if and only if $a\breve{K}h$ is true:

$$hKa \leftrightarrow a\breve{K}h. \tag{37}$$

However, not only h, but also i and j are covered by h. A binary relation S is symmetric if "y is in the relation S to z" implies "z is in the relation S to y" and vice versa:

$$ySz \leftrightarrow zSy.$$

In view of (37), one may say that a symmetric relation is "self-converse," i.e., conversion leaves a symmetric relation unchanged.

Piaget (1972, p. 339) maintains "... that the reciprocity R corresponds to the inverse operation ... of relations (and to the permutation of the ... implication between propositions)." He also asserts that "the binary operations include only one transformation by permutation [that transforms $p \to q$ into $q \to p$] and vice versa ..., namely, $q \to p = R(p \to q)$" (Piaget, 1952, p. 48).

Without any further restriction, there are, of course, 14! such transformations. If, however, we take into account only automorphisms of FB(2), then Table 2 shows that *there are two of them* which transform $p \to q$ into $q \to p$ and vice versa, that is, c into b, namely Piaget's R and the automorphism which we have labelled T. There is no d-automorphism that does the job. In fact, it is T that corresponds to the operation of conversion of a relation. This is plausible, since conversion leaves symmetric relations unchanged and since T is the automorphism that fixes the eight symmetric elements of FB(2), namely, $\{v, a, d, e, j, k, x, o\}$.

Presently, we shall elaborate this assertion and qualify it. But first we have to recall the algebraic structure of relations on a set of elements. Detailed information may be found, for instance, in Tarski (1941) and in Jónnson and Tarski (1951 and 1952).

On the set A of binary relations on a finite set, it is possible to define Boolean (or absolute) operations: two binary operations $\wedge$ and $\vee$ (meet and join), one unary operation $'$ (complement) and two special relations 0 and 1 (null relation and universal relation) such that the algebra

$$\mathscr{B} = \langle A, \wedge, \vee, ', 0, 1\rangle$$

is a Boolean algebra, the "Boolean algebra of relations." The binary relations on a set B of m elements may be interpreted as subsets of the set $B \times B$ with m^2 elements. (Finer distinctions are possible and necessary in other contexts.) Then the operations meet, join, and complement of $\mathscr{B}$ simply are the set-theoretic operations of intersection, union, and complement. Thus, the algebra $\mathscr{B}(m)$ of binary relations defined on a set of m elements is isomorphic to a Boolean algebra $\mathrm{B}(m^2)$ with 2^q $(q = m^2)$ elements. On the other hand, the free Boolean algebra $\mathrm{FB}(n)$ of n-ary propositional operations is, as we know, isomorphic to $\mathrm{B}(2^n)$. Putting $n = 2p$, we find the cases in which the Boolean algebra of relations is isomorphic to a free Boolean algebra (of propositional operations):

$$\mathscr{B}(2^p) \simeq \mathrm{FB}(2p),$$

i.e., the Boolean algebra of relations on a set of 2^p elements is isomorphic to the free Boolean algebra with $2p$ generators. Thus, the relations $\mathscr{B}(2)$ on a set of two elements are isomorphic to $\mathrm{FB}(2)$; $\mathscr{B}(3)$ is not isomorphic to any *free* Boolean algebra, in particular not to $\mathrm{FB}(3)$, but $\mathscr{B}(4)$ is isomorphic to $\mathrm{FB}(4)$, $\mathscr{B}(8)$ to $\mathrm{FB}(6)$, and so on.

So far, the—Boolean—algebra of relations has not provided us with any means of reducing the number of automorphisms (and d-automorphisms) which should be taken into account. However, operations other than the Boolean operations can be defined on the set of binary relations.

These operations are termed *Peircean* or relative operations by Tarski. There are two binary operations, the multiplication $\cdot$ and the addition $+$, one unary operation, the conversion $\breve{}$, and two special relations, the identity relation E and the diversity relation J, in addition to complementation and to the relations 0 and 1 of the Boolean structure that play a role also in the Peircean structure. More precisely, we can say that this structure, the *Peircean algebra of relations* is given by

$$\mathscr{P} = \langle A, \cdot, +, \breve{}, ', E, J, 0, 1\rangle,$$

where the operations have the special properties indicated below.

(i) $A_{\cdot} = \langle A, \cdot, E, 0\rangle$ is a monoid (unit element E) with null element 0.

(ii) $A_{+} = \langle A, +, J, 1\rangle$ is a monoid (unit element J) with null element 1.
(Note: addition, like multiplication, of relations is not commutative !)

(iii) The unary operation $\breve{}$ is characterized by:

$$(X)\breve{}\breve{} = X, \tag{38}$$

$$(X \cdot Y)\breve{} = \breve{Y} \cdot \breve{X}, \tag{39}$$

$$(X + Y)\breve{} = \breve{Y} + \breve{X}, \tag{40}$$

i.e., $\breve{}$ is an idempotent *anti*-automorphism (*a*-automorphism) of each of the monoids $A_.$ and A_+. ("Anti-," because the order of the elements is inversed.)

(iv) The unary operation $'$, in addition to its Boolean properties, obeys identities of the De Morgan type:

$$(X \cdot Y)' = X' + Y', \tag{41}$$

$$(X + Y)' = X' \cdot Y', \tag{42}$$

i.e., $'$ is an isomorphism of the monoids $A_.$ and A_+. By analogy with the Boolean case, we say also that $'$ is a *d*-automorphism of $\mathscr{P}$. Moreover, the Boolean and the Peircean structures are linked by the distributivity (right and left) of $\cdot$ with respect to $\vee$ and by the distributivity (right and left) of $+$ with respect to $\wedge$.

Again, if a relation is interpreted as a subset, then the following set-theoretic interpretation of the Peircean operations can be given:

—$(y, z) \in X \cdot Y$ means that there exists an element u such that $(y, u) \in X$ and $(u, z) \in Y$;

—$(y, z) \in X + Y$ means that for all u one has either $(y, u) \in X$ or $(u, z) \in Y$ or both;

—$(y, z) \in \breve{X}$ means that $(z, y) \in X$.

Furthermore:

—$(y, z) \in X'$ means the we have *not* $(y, z) \in X$.

We write from now on N for $'$. The next step is to establish carefully that one can write T for $\breve{}$.

We now substantiate and make more precise our claim that the converse of a relation X is the relation TX and that it cannot be RX. We shall first say what is meant by symmetric BPO. A BPO, i.e., an element of FB(2), is symmetric if it is left unchanged by the exchange of the two generators chosen to express the elements of FB(2). This definition obviously depends on the choice of generators—and not only on what we have termed the generator system. Thus, for what may justifiably be called the "standard generators," viz. (p, q) in that order, the eight symmetric BPO are $\{v, a, d, e, j, k, x, o\}$ the "standard symmetric BPO." It is the automorphism T that leaves this set invariant elementwise. These are the symmetric BPO also for the generators (q, p), (p', q'), and (q', p'). The choices (p, q'), (q', p), (p', q), and (q, p'), single out the BPO $\{v, b, c, e, j, l, m, o\}$ as symmetric, and the automorphism TR fixes each of these. Similarly, the two other generator systems give rise to two sets of symmetric BPO. (Cf. columns 4 and 5 of Table 1.) This becomes almost obvious in the crystallographic representation of Appendix 1.

As we apply the transformations of $\mathscr{M}_2$ to the 16 BPO, the standard generators successively transform into each of the 24 possible generator pairs, the set of standard symmetric BPO is successively transformed into each of the six sets of symmetric BPO, and the subgroups that fix the symmetric BPO are transformed at the same time. If a transformation $\tau \in \mathscr{M}_2$ transforms a symmetric BPO, say s into τs and if the transformation ϕ fixes s ($\phi s = s$), then obviously, $\tau\phi\tau^{-1}$ (the conjugate of ϕ by τ) fixes τs. Thus, as τ varies over the group $\mathscr{M}_2$, the automorphism T that fixes the standard symmetric BPO, is transformed successively into all elements of its conjugate class. Tables 2 and 3 are divided into conjugate classes. As can be seen, Piaget's R does not belong to the conjugate class of T. Thus, it cannot be, for any choice of generators, the transformation that fixes the symmetric BPO and transforms the others into their symmetric counterpart.

There remains to transfer these findings from the free Boolean algebra of the 16 BPO to the isomorphic algebra of binary relations on a set of two elements. To this end, we remark that the choice of an ordered pair of generators—(p, q) in our case—enables us to regard the elements of FB(2) as *Boolean functions*. Let 2 be the set $\{1, 0\}$ and **2** this set ordered by $o \leqslant 1$. A Boolean function ϕ is a function on the n-fold direct product $2 \times \cdots \times 2$ (domain) to **2** (codomain):

$$\phi: 2 \times \cdots \times 2 \to \mathbf{2}.$$

We write $\mathbf{2}^{2\cdots 2}$ (with n times 2 in the exponent) for the set of all n-ary Boolean functions.

It is now possible to identify elements of FB(n) with such functions. For instance $a \in$ FB(2) is identified with the function that has the value 0 at the ordered pair (0, 0) and the value 1 for the other three pairs (because $p \vee q$ is false when both p and q are false, and true in the other three cases). Similarly, the value of $m \in \mathbf{2}^{22}$ at (0, 1) is 1 and it is 0 at the other arguments. This establishes an isomorphism

$$\text{FB}(2) \simeq \mathbf{2}^{22}. \tag{43}$$

We shall use the same symbols for the elements of FB(2) and the corresponding ones of $\mathbf{2}^{22}$.

In the preceding section, we had chosen a generator system and then found the invariance group G_1 of that system. It is clear that, essentially, what we have found there is the group of mixed automorphisms of the Boolean functions $\mathbf{2}^{22}$. The subgroup of automorphisms is isomorphic to the automorphisms of the domain of these functions, i.e., of the direct product 2×2; this group is isomorphic to D_4, the d-automorphisms are obtained by adjoining to these the (only) d-automorphism of the codomain **2**.

Remember, however, that the isomorphism (43) is based on the choice of a definite pair of generators: the standard pair (p, q). It may, therefore, be termed the "standard isomorphism." But if, for instance, we choose the generators (p, e), then it would be the element $p \vee q'$ that corresponds to the Boolean function a, and $p' \wedge q'$ would correspond to m.

Exactly in the same way, i.e., by a choice of generators of FB(n), it is possible to establish an isomorphism

$$\text{FB}(n) \simeq \mathbf{2}^{2\cdots 2}. \tag{44}$$

The next step is to provide an isomorphism

$$\mathbf{2}^{22} \simeq \mathscr{B}(2). \tag{45}$$

A convenient way is to identify the Boolean function $\phi \in \mathbf{2}^{22}$ with the relation ϕ defined by

$$(y, z) \in \phi \qquad \text{if and only if} \quad \phi(y, z) = 1$$

(i.e., to consider the Boolean function as the characteristic function of a subset). This establishes a one-to-one mapping between $\mathbf{2}^{22}$ and $\mathscr{B}(2)$. We shall use the same symbols for the elements thus related. By the same token, the automorphisms of FB(2) now represent isomorphisms between FB(2) and $\mathscr{B}(2)$; the standard isomorphism corresponds to the identity automorphism.

We have defined what is meant by symmetric BPO. It is clear that this definition carries over to binary Boolean functions and to binary relations. A binary Boolean function ϕ is symmetric if, for any y, $z \in 2$,

$$\phi(z, y) = \phi(y, z).$$

A binary relation ϕ is symmetric if, for any y, z,

$$(z, y) \in \phi \qquad \text{implies} \quad (y, z) \in \phi.$$

This shows: (i) that with the standard isomorphism, it is T that leaves symmetric relations invariant, and (ii) that the operation $\breve{\ }$ may be identified with the transformation T:

$$TX = \breve{X}. \tag{46}$$

Concerning the role of automorphisms, our previous discussion of symmetric BPO carries over too. In any case, the transformation that leaves invariant a symmetric relation, must belong to the conjugate class of T; therefore, it cannot be R. Furthermore, from (39), (40), and (46), we know already that T is not an automorphism of the Peircean structure; it is an anti-automorphism of $\mathscr{P}$. Which, then, of the Boolean automorphisms are also Peircean ones?

The only automorphisms with respect to the Peircean structure of relations on a finite set come from renumberings of the elements of the set. Thus, the Peircean automorphisms of $\mathscr{P}(n)$ form the group S_n of all permutations of n objects. For $\mathscr{P}(2)$, this is the group of the order two; the automorphism itself is easily recognized as Piaget's R, in case the standard isomorphism is chosen. For any choice of isomorphism, this automorphism is an element of the conjugate class of R, which is $\{P, Q, R\}$. Thus, R does not correspond to the "inverse operation of relations," but is the only Peircean automorphism of $\mathscr{P}(2)$, in addition, of course, to the identity.

If we take into account the Boolean and the Peircean structure of relations, we have

$$\mathscr{R} = \langle A, \wedge, \vee, \cdot, +, N, T, 0, 1, E, J \rangle$$

the "*algebra of relations.*" We are interested in the structure-preserving transformations of $\mathscr{R}$. We have met four types of transformations.

(i) Transformations that are automorphisms of $\mathscr{B}$ and of $\mathscr{P}$. We have seen that R is an automorphism of $\mathscr{B}(2)$ and of $\mathscr{P}(2)$. Such transformations will be termed *automorphisms* of $\mathscr{R}$.

(ii) Transformations that are d-automorphisms of $\mathscr{B}$ and of $\mathscr{P}$, for instance, N and $RN = \bar{R}$ are d-automorphisms of $\mathscr{B}(2)$ and of $\mathscr{P}(2)$. They are the *d-automorphisms* of $\mathscr{R}$.

(iii) Transformations that are automorphisms of $\mathscr{B}$ and a-automorphisms of $\mathscr{P}$. Since the binary operations of $\mathscr{B}$ are commutative, there are no a-automorphisms of $\mathscr{B}$. We call these the *a-automorphisms* of $\mathscr{R}$. Example: T and TR.

(iv) Transformations that are d-automorphisms of $\mathscr{B}$ but are dual a-automorphisms with respect to $\mathscr{P}$. These are the *da-automorphisms* of $\mathscr{R}$. The transformations TN and $T\bar{R}$ are da-automorphisms of $\mathscr{R}(2)$:

$$TN(r \cdot s) = TNs + TNr. \tag{47}$$

It is easily seen that T and N—in addition to commuting one with the other—also commute with all automorphisms of $\mathscr{P}(n)$. Furthermore, T and N cannot be obtained by any renumbering. Thus, the group $\mathscr{G}_n$ of all automorphisms, d-automorphisms, a-automorphisms, and da-automorphisms of $\mathscr{R}(n)$ is a direct product

$$\mathscr{G}_n = S_n \times \langle T \rangle \times \langle N \rangle \simeq S_n \times D_2, \tag{48}$$

where D_2 is generated by T and N. In particular,

$$\mathscr{G}_2 = \langle R \rangle \times \langle T \rangle \times \langle N \rangle, \tag{49}$$

a group we have already met as G_3 in the preceding section (on subgroups).

Any subgroup L of $\mathscr{G}_n$ that contains automorphisms, a-automorphisms, d-automorphisms, and da-automorphisms, contains the same number of each sort. The order of such a group is a multiple of four. In general, however, such groups are not direct products of the subgroup of automorphisms L' with D_2; but always

$$L/L' \simeq D_2 \tag{50}$$

(i.e., L is an extension of L' with D_2). Of course, also

$$\mathscr{G}_n/S_n \simeq D_2. \tag{51}$$

In conclusion, *a group isomorphic* to D_2 does appear in quite an essential way whenever one considers structure preserving transformations of algebras of relations. This group, however, is not INRC but $\{I, T; N, TN\}$.

CONCLUDING REMARKS

We have discussed in some detail some of the questions concerning transformations of propositional operations. It is, of course, impossible to be complete within the limits of a paper, even in such a limited field. But whatever the coverage may be, care of details is necessary and rewarding; the link with experiment is at this level and at this price. Here we were specially interested in the gradual building-up of groups of structure preserving transformations from involutions, and in the manner by which transformations other than involutions come about through this process. The paper should also illustrate the kind of reflection that could be important in modeling the algebra of competence.

APPENDIX 1: CRYSTALLOGRAPHIC REPRESENTATION OF $\mathscr{M}_n$

In this appendix, we advance a representation of $\mathscr{M}_2$ as a group of all spatial transformations of a cube. Our purpose is not simply to present a curiosity, nor is it only to provide a convenient way of visualizing logical transformations as spatial ones. The isomorphism between $\mathscr{M}_2$ and the symmetries of a cube raises the question of a *possible correlation* (possibly delayed) between the utilization of a group of logical transformations and the mastery of an isomorphic group of spatial transformations. Therefore knowledge of such transformations seems appropriate.

The facts are the following. The group $\mathscr{M}_n$ of mixed automorphisms of the free Boolean algebra with n generators FB(n) is isomorphic to the symmetry group of a $(2^n - 1)$-dimensional crystal. In the case of FB(2), the Boolean algebra of the sixteen BPO, the crystal is an ordinary, that is, 3-dimensional, crystal. The group $\mathscr{M}_2$ is then the group of all spatial symmetries of a cube. (In crystallographic notation $\mathscr{M}_2 \simeq m\bar{3}m$, $\mathscr{A}_2 \simeq \bar{4}3m$.)

Therefore, it is easy to represent the elements of $\mathscr{M}_2$ by spatial transformations. We only need to place the elements of FB(2) suitably on the eight vertices and the six centers of the faces of a cube, e.g., as shown in Fig. 3. This is the configuration of the face-centered cube, well known to crystallographers. Two of the 16 BPO are missing,

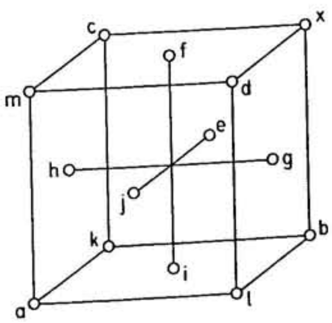

FIG. 3. Cubic symmetry of binary propositional operations.

viz. the tautology v and the antilogy o. These elements undergo only two transformations: either they remain unchanged (under automorphisms) or they are exchanged (by d-automorphisms).

To each element of $\mathscr{M}_2$ there corresponds now a symmetry operation of the cube, as shown in Table 6. The elements of $\mathscr{M}_2$ are given by the products found in the column "Generators" of Tables 2 and 3, the transformations of the cube by their international crystallographic symbol to which subscripts have been added that specify their orientation. The position of these symmetry elements with respect to the cube, is shown in Fig. 4; the meaning of the symbols is explained in the following remarks.

Let us first observe that the transformation N (already termed "inverse" by Piaget) corresponds to inversion $\bar{1}$ through the centre O of the cube. All axes of Fig. 4 contain O. Three of them, x, y, and z, pass through the centers of opposite faces. About these axes there are rotations by 180° (such as, 2_x), rotations by 90° (4_x), or by 270° (4_x^3). Furthermore, these transformations combined with the inversion $\bar{1}$ also

TABLE 6

Crystallographic Interpretation of $\mathscr{M}_2$ as $m\bar{3}m$

$\mathscr{A}_2$	$\bar{4}3m$		
I	1	N	$\bar{1}$
VP	m_F	$V\bar{P}$	2_F
U	m_D	UN	2_D
TR	m_A	$T\bar{R}$	2_A
T	m_B	TN	2_B
UQ	m_C	$U\bar{Q}$	2_C
V	m_E	VN	2_E
P	2_z	$\bar{P}$	m_z
Q	2_y	$\bar{Q}$	m_y
R	2_x	$\bar{R}$	m_x
TU	3_δ	TUN	$\bar{3}_\delta$
$TUVT$	3_γ^2	$TUVTN$	$\bar{3}_\gamma^5$
UT	3_δ^2	UTN	$\bar{3}_\delta^5$
UV	3_β	UVN	$\bar{3}_\beta$
$TVUT$	3_γ	$TVUTN$	$\bar{3}_\gamma$
VU	3_β^2	VUN	$\bar{3}_\beta^5$
TV	3_α^2	TVN	$\bar{3}_\alpha^5$
VT	3_α	VTN	$\bar{3}_\alpha$
UP	$\bar{4}_y^3$	$\bar{R}U$	4_y^3
TQ	$\bar{4}_x^3$	$\bar{P}T$	4_x^3
VR	$\bar{4}_z$	$\bar{Q}V$	4_z
QT	$\bar{4}_x$	$T\bar{P}$	4_x
RV	$\bar{4}_z^3$	$V\bar{Q}$	4_z^3
PU	$\bar{4}_y$	$U\bar{R}$	4_y

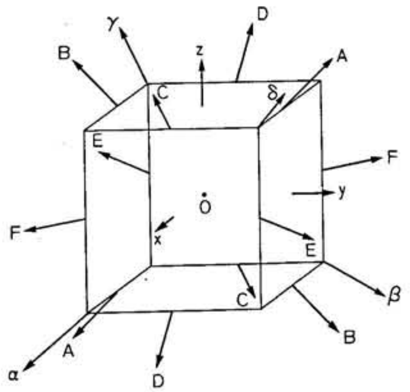

FIG. 4. Symmetry elements of the cube.

occur. The combination $2_x\bar{1}$ is shortened as m_x; it represents reflexion through a plane, a mirror, perpendicular to the x axis (and passing through O). The combination $4_x\bar{1}$ is written $\bar{4}_x$ and is termed a rotatory inversion.

Four axes, α, β, γ, and δ, pass through the vertices of the cube. Around these axes there are rotations by 120° (3_δ) and by 240° (3_δ^2) as well as the corresponding rotatory inversions ($\bar{3}_\delta$, $\overline{3_\delta^2} = \bar{3}_\delta^5$).

Positive and negative rotations about these two families of axes are distinguished; therefore they are fitted with one arrowhead. There are, however, six axes, A, B, C, D, E, and F, fitted with two arrowheads around which only rotations of 180° (2_A) and reflexions (m_A) through planes perpendicular to them (and passing through O) occur.

Remember now that the standard symmetric BPO are $\{v, a, d, e, j, k, x, o\}$. Except for v and o, which are not represented in the figure, they are situated in the plane corresponding to m_B, and, of course, they are left unchanged by reflexion in that plane, that is by $T = m_B$. The symmetric BPO with respect to the generators (p, q') are located in the plane corresponding to m_A and are fixed by $m_A = TR$. These planes intersect in the x axis (the A axis and the B axis are perpendicular to the x axis). The x axis O_x is distinctive of the generator system $\{p, q\}$ to which the generators (p, q) and (p, q') belong. The group G_1 preserving this sytem, detailed in Table 5, singles out the x axis; it is the symmetry group of a square prism, with the square perpendicular to the x axis ($G_1 \simeq 4_x/mmm$, $G_1' \simeq \bar{4}_x 2_y m$). The other two generator systems correspond to the other two directions parallel to the edges of the cube: $\{i, e\}$ to O_y and $\{e, h\}$ to O_z.

Many other features discussed in the paper may be read off from the Figs. 3 and 4. We cannot detail them all. Let us just mention the following. Not only the action of an element τ of $\mathscr{M}_2$ on the BPO may easily be found from Fig. 3, but also the conjugate by τ of an element ϕ of $\mathscr{M}_2$, that is $\phi\tau\phi^{-1}$. It is obtained by performing on Fig. 4 the geometric transformation corresponding to τ. For instance, $3_\delta m_C 3_\delta^{-1} = m_E$ and indeed performing a rotation of 120° around the axis δ takes the C-axis to the place of the E-axis. This is easier to do than to compute directly $(TU)(UP)(UT)$.

APPENDIX 2: More about the "Restricted Groups"

The group G_1', the restricted group of automorphisms of the section on subgroups, a group isomorphic to D_4, has already been found by Pólya (1937 and 1940). He posed and solved a logical problem whose origin he traces to Jevons. The first step towards the solution, the only one in which we are interested here, is to find the "hypercubic" groups K_n formed by the operations of permuting and/or complementing the n variables of n-ary Boolean functions. The 2^n possible values of the argument of such a function can be placed on the vertices of a hypercube, that is a cube in n dimensions. For $n=2$ this is a square; for $n=3$ it is the ordinary cube as shown in Fig. 5. Note that this utilization of the cube is different from that in Appendix 1.

The structure of K_n has been determined by Pólya as "*wreath product*" of the group of two elements C_2 by the group S_n of all permutations of n objects. This wreath product is a semi-direct product of $C_2 \times \cdots \times C_2$ (n-times) by S_n,

$$(C_2 \times \cdots \times C_2) \dot{\times} S_n, \tag{52}$$

where an element s of S_n acts on $C_2 \times \cdots \times C_2$ by permuting the n generators of the n copies of C_2. Therefore, Pólya (1937) uses the pictorial expression "C_2-wreath around S_n."

The group K_n has $n!2^n$ elements. For unary, binary, and ternary Boolean functions we have the following isomorphisms

$$K_1 \simeq C_2 \tag{53}$$

$$K_2 \simeq D_4 \simeq G_1' \tag{54}$$

$$K_3 \simeq S_4 \times C_2. \tag{55}$$

Thus K_3, the group of *automorphisms* of the *ternary* Boolean functions $2^{2\times 2\times 2}$, which is also the *restricted* group of automorphisms of FB(3), is isomorphic to the group of *all mixed* automorphisms of the *binary* propositional operations FB(2). Therefore, we should be able to read the transformations of Tables 2 and 3 as giving all transformations of commuting and/or complementing the three standard generators, say p, q, and r, of FB(3). (There are of course many other choices.) This is indeed easily done. We restrict the transformations in question to the set $\{i, f, h, g, e, j\}$ and we identify these elements with p, q', q, q', r, and r', respectively. Note, however, that in our former interpretation as elements of FB(2), the elements e and j were not independent of i, f, h, and g; now as generators of FB(3), they are independent, of course. Placed at the centres of the faces of the cube (as in Fig. 5), they form a 3-dimensional octahedron; in general we find an n-dimensional octahedron.

Let us now examine how the restricted groups may be successively built up as we

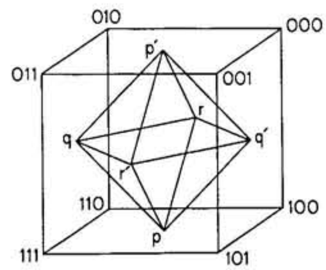

FIG. 5. The restricted group K_3 of automorphisms of FB(3).

add one generator at a time to a free Boolean algebra. For $n=1$ we have as single generator of $K_1 = C_2$ the permutation (pp') which we note $\bar{P}$:

$$\bar{P} = (pp'). \tag{56}$$

(Inspection of Tables 2 and 3 shows that there are eight elements of $\mathscr{M}_2$ which contain $(pp') = (if)$. The reason for the choice of $\bar{P}$ will become clear, presently.) For $n=2$ we *add* as second generator of

$$K_2 = (C_2 \times C_2) \dot{\times} S_2$$

the element

$$T = (pq)(p'q') \tag{57}$$

the generator of S_2; it permutes the two selected generators of FB(2). (In fact there are two elements of $\mathscr{M}_2$ that contain these permutations.) We then find

$$T\bar{P}T = \bar{Q} = (qq'). \tag{58}$$

For

$$K_3 = (C_2 \times C_2 \times C_2) \dot{\times} S_3,$$

we have to *add* a generator that, together with T generates S_3, We can take

$$U = (pr)(p'r'). \tag{59}$$

Then

$$U\bar{P}U = \bar{R} = (rr'), \tag{60}$$

and so on for higher values of n. Thus *the number of generators of the group K_n of restricted automorphisms of* FB(n) *is n.*

Incidentally, we have thus generated (as mentioned in the section on dual automorphisms), $\mathscr{M}_2 \simeq S_4 \times C_2$ by the three involutions $\bar{P}$, T, and U. However, thus generated, some apparently simple operations get complicated expressions. For instance,

$$N = \bar{P}T\bar{P}U\bar{P}UT. \tag{61}$$

Such *complicated generation should be avoided, unless it reflects genetical complexities.*

More recently, Leresche (1978) set out to determine "the complete group of binary [propositional] operations." By binary operations he understands the 10 (out of 16) binary operations, that actually depend on two arguments, p and q. In our notation, these are the 10 elements of the set $\{a, b, c, d, e, j, k, l, m, x\} = L$.

From the preceding it is not quite clear, however, what kind of transformations should be selected out of the 10! candidates since the set L has no structure. Indeed, it is not closed under meet and/or join. If then, to remedy, one adds the elements v and o, the distributivity equations (9) and (10) do not hold in general, and complementation is not unique. The set in question, even augmented, is not a Boolean algebra and, a fortiori, not a Boolean subalgebra of FB(2).

The way out is to say that one is interested only in those transformations belonging to $\mathscr{M}_2$ that transform the elements of L among themselves. Consequently, these transformations transform the remaining 6 elements of FB(2) among themselves. Since the elements v and o always keep apart, this amounts to looking for those transformations of $\mathscr{M}_2$ that permute the four elements i, h, g, and f among themselves. But this is exactly the program that in the section on subgroups led to the group G_1 of Table 5, whose subgroup of automorphisms G_1' is the group K_2 already found by Pólya. However, the "complete group of transformations" found by Leresche, in his otherwise correct paper, is only the Boolean subgroup G_2 of G_1, whose elements are given in (35). What is missing is the transformation T that exchanges the two generators. Similarly, for ternary propositional operations, he finds a Boolean group of order 16 instead of $K_3 \times C_2 \simeq S_4 \times D_2$ which is of order 96.

APPENDIX 3: Bart's Generalization Model

In an interesting paper, Bart (1971) has chosen another way to restrict the abundance of the symmetric group S_{16}. He requires two things of the transformation group:

(i) Every element of the group should be an involution. Such groups are sometimes termed Boolean groups.

(ii) The group should be a regular permutation group. A group G of permutations on a set S is *transitive* if for any $x, y \in S$ there exists an element $\alpha \in G$, such that $\alpha x = y$, i.e., G has a single orbit on S. Such a group is *regular*, if no element of it, except, of course, the unit element leaves any elements of S unchanged. If S is finite then the number of elements of a regular group G (the order of G) equals the number of elements of S (the degree of G). A commutative and transitive group of permutations is regular.

Taken separately, the two conditions are not very restrictive; there are many subgroups of S_{16} that fulfil them. But taken together, do they seriously limit the possibilities? Bart's procedure for obtaining his group seems rather natural. It may be formulated as follows:

(i) From the elements FB(2), one forms a *ring* by defining addition by

$$x + y = xwy, \tag{62}$$

and multiplication by $xy = x \wedge y$. (This is Stone's construction.)

(ii) With this addition, the elements of FB(n) constitute a Boolean group H_n with unit element the zero, 0 of FB(n). (This is why the group is termed Boolean.) Further it has by construction the right number of elements, namely 2^p ($p = 2^n$).

(iii) A natural way to define the action ϕ_ξ of an element ξ of H_n on an element x of FB(n) is to define

$$\phi_\xi(x) = \xi + x. \tag{63}$$

Due to the group properties of H_n, the set

$$\Phi_n = \{\phi_\xi : \xi \in K_n\} \tag{64}$$

is a regular Boolean group of transformations of FB(n), and

$$\phi_\xi \phi_\eta = \phi_{\xi+\eta}. \tag{65}$$

The dual construction of a Boolean ring by

$$x +' y = x \leftrightarrow y, \qquad x \cdot y = x \vee y \tag{66}$$

gives rise to transformations ϕ'_ξ defined by

$$\phi'_\xi(x) = x +' y. \tag{67}$$

The group obtained this way is again Φ_n, because

$$\phi'_\xi = \phi_{\xi'}, \tag{68}$$

where ξ' is the complement, in FB(n), of ξ.

Bart's group Φ_2 is shown in Table 7. It will be noticed that $I = \phi_o$ and $N = \phi_v$ are the only elements of $\mathscr{M}_2$ that are also elements of Φ_2.

However, for $n > 1$, Φ_n is not the only regular Boolean transformation group of the elements of FB(n) $\cdot$ Φ_n is not a normal subgroup of S_q, where q ($q = 2^p$, $p = 2^n$) is the number of elements of FB(n); the elements of Φ_n do not form a complete conjugate class of S_q. (For $n = 2$, they belong to a class with roughly two million

TABLE 7

Bart's Group of Transformations of BL(2)

ξ	ϕ_ξ
v	$(vo)(ax)(bm)(cl)(dk)(if)(hg)(ej)$
a	$(vx)(ao)(bf)(cg)(de)(im)(hl)(jk)$
b	$(vm)(af)(bo)(cj)(dh)(ix)(el)(gk)$
c	$(vl)(ag)(bj)(co)(di)(hx)(em)(fk)$
d	$(vk)(ae)(bh)(ci)(do)(jx)(gm)(fl)$
i	$(vf)(am)(bx)(cd)(io)(hj)(eg)(kl)$
h	$(vg)(al)(bd)(cx)(ij)(ho)(ef)(km)$
e	$(vj)(ad)(bl)(cm)(ig)(hf)(eo)(kx)$
j	$(ve)(ak)(bc)(dx)(ih)(jo)(gf)(lm)$
g	$(vh)(ac)(bk)(dm)(ie)(jf)(go)(lx)$
f	$(vi)(ab)(ck)(dl)(he)(jg)(fo)(mx)$
k	$(vd)(aj)(bg)(cf)(il)(hm)(ex)(ko)$
l	$(vc)(ah)(be)(df)(ik)(jm)(gx)(lo)$
m	$(vb)(ai)(ce)(dg)(hk)(jl)(fx)(mo)$
x	$(va)(bi)(ch)(dj)(ek)(gl)(fm)(xo)$
o	$(v)(a)(b)(c)(d)(i)(h)(e)(j)(g)(f)(k)(l)(m)(x)(o).$

elements.) All the (numerous) subgroups of S_q conjugate to Φ_n are regular Boolean transformation groups of the elements of FB(n). (Furthermore, they arise from commutative binary operations on the set of elements of FB(n), in the same way Φ_n arises from the operation w in (64).)

We have now shown how to obtain a great number of regular Boolean transformations on the set of n-ary propositional operations. It has, however, been argued in the text, that if we model reversibility in cognitive activity by operations α having an inverse α^{-1} (as understood in group theory), the further requirement that $\alpha^{-1} = \alpha$, termed the "strong view of reversibility" (Bart, 1971, p. 542) is too strong. If involutions play a role, then the experience of their possible noncommutativity leading to higher order operations is necessary for later stages of cognitive activity. As regards transitivity, Piaget was conscious of the fact that INRC is not transitive, and was not at all bothered by it. Quite to the contrary, transitivity as well seems to be too strong a condition. Indeed, according to the interpretation of orbits proposed in this paper, transitivity would mean that a person would go through all the elements of the combinatorial manifold, which for logical operations at the stage of formal thought is taken by Piaget to be FB(n) without bothering too much about the value of n.

Bart's basic postulate is precisely "that an individual may be characterized by FB(n), where n is the maximal number of factors for which he can generate all the possible combinations. Thus, for an individual FB(n) would be an indicant of the level of formal operational capacity and would state the maximal number of factors n for which the individual can generate all the possible combinations" (Bart, 1971, p. 540).

It is then natural for him to require that the set $[\Pi_n]$ of 2^n generators of the group Φ_n acting on FB(n) should "form an inclusion chain

$$\Pi_1 \subset \Pi_2 \subset \cdots \subset \Pi_n \cdots"$$

(Bart, 1971, p. 545). Note, however, that at each step 2^n generators are added. A further feature, which Bart emphasizes in discussing an example from Inhelder and Piaget (1958), is that "with the proposed formal thought model ... the formal transformation that allows an individual to consider a proposition of the form $q \rightarrow r$ after considering a proposition of the form $p \rightarrow r$ can be designed."

We want to point out, that both these features are present already in Pólya's groups K_n (and a fortiori in $\mathscr{M}_n$). Progressing from FB(n) to FB($n+1$) by adding one new generator p_{n+1}, to the n old ones $\{p_i: 1 \leqslant i \leqslant n\}$, we have simply to add one generator

$$g_{n+1} = (p_1 p_{n+1})(p'_1 p'_{n+1})(p_2)(p'_2) \cdots (p_n)(p'_n)$$

to the n old ones. Furthermore, one sees from (57) that K_3 contains the transformation needed to transform $p \rightarrow r$ into $q \rightarrow r$, namely,

$$T(p \rightarrow r) = (q \rightarrow r).$$

We therefore think that the groups $K_n \times C_2$ are very convenient for the description of formal operations.

Acknowledgments

This work was supported by the *Fonds National Suisse de la recherche scientifique* (project 1.345–0.81). I have profited from discussion with Gil Henriques and I enjoy the hospitality of the Department of Theoretical Physics of the University of Geneva, Switzerland. Furthermore, I want to thank a referee for a careful reading of the manuscript and numerous valuable suggestions.

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RECEIVED: June 1, 1983