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On Computing the Polynomial Invariants of a Finite Group

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Let ρ be a real representation of a finite group $\mathbf{G}$ as $n \times n$ matrices and $P(\rho)^{\mathbf{G}}$ the ring of polynomial invariants associated with $\rho(\mathbf{G})$. One way to describe $P(\rho)^{\mathbf{G}}$ is as a direct sum $\bigoplus_{i=0}^{d} t_i \mathbb{R}[f_1, \ldots, f_n]$. Given that such a *good polynomial basis* $f_1, \ldots, f_n$, $t_0, \ldots, t_d$ is known for $P(\rho)^{\mathbf{G}}$, we will show how to construct good polynomial bases for other polynomial rings associated with $P(\rho)^{\mathbf{G}}$: $P(\rho)^{\mathbf{H}}$ where $\mathbf{H}$ is a subgroup of $\mathbf{G}$, $P(\rho \oplus \sigma)^{\mathbf{G}}$ where σ is another real representation of $\mathbf{G}$, and $P(\bigoplus^m \rho)^{\mathbf{G}}$. We will make sense of the notion of *good polynomial basis for relative invariants* and show how to construct the same for the representation $\bigoplus^m \mu_i \rho$, where $\mu_i \rho$ is the representation gotten from ρ by *twisting* it by the linear representation μ_i, $i = 1, \ldots, m$.

If $P(\rho)$ is the ring of all polynomials associated with $\rho(\mathbf{G})$, then those features of the structure of $P(\rho)$ as a graded $\mathbf{G}$-algebra—needed for the constructions above—will also be developed by extending classical results about the ideal in $P(\rho)$ generated by the invariants, about $\mathbf{G}$-harmonic polynomials and about polarization.

1. INTRODUCTION

In this paper the development of tools for computing invariants was motivated by a desire to find the invariants associated with an arbitrary real representation of any abstract three-dimensional crystallographic point group. This class of 17 finite groups is particularly nice. The irreducible representations are all low-dimensional and have image groups that are either reflections groups or are closely related to reflection groups, whose invariants are known and well behaved. We

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want to capitalize on this nice situation to describe the invariants of all the representations in question.

The techniques developed in this paper are valid for and may be used with arbitrary finite groups. However, they were designed for representations whose irreducible constituents have the nice characteristics mentioned above. They may not be computationally useful for other classes of representations of groups (e.g. those with "messy invariants' studied by Huffman and Sloane [7]).

In this paper, the notion of good polynomial basis will be the primary means for describing the structure of the polynomial invariants associated with a representation. We recall that this notion is defined as follows.

Let $\mathbf{G}$ be a finite group of real $n \times n$ matrices. If $p(X_1, \ldots, X_n)$ is a polynomial in the variables $X_1, \ldots, X_n$ with real coefficients and $\mathbf{g} = (a_{ij})$ is an element of $\mathbf{G}$, then we let $\mathbf{G}$ act on $P = \mathbb{R}[X_1, \ldots, X_n]$ by $\mathbf{g}p(X_1, \ldots, X_n) = p(\sum a_{j1} X_j, \ldots, \sum a_{jn} X_j)$. The ring $P^{\mathbf{G}}$ of *invariants* of $\mathbf{G}$ consists of all polynomials p such that $\mathbf{g}p = p$ for all $\mathbf{g}$ in $\mathbf{G}$. A convenient way of describing $P^{\mathbf{G}}$ is as

$$P^{\mathbf{G}} = \bigoplus_{k=0}^{d} t_k \mathbb{R}[f_1, \ldots, f_n]$$

where $f_1, \ldots, f_n$ are algebraically independent homogeneous polynomials, $t_0 = 1$ and $t_1, \ldots, t_d$ are other homogeneous invariants. The set of invariants $f_1, \ldots, f_n, t_1, \ldots, t_d$ form what is called a *good polynomial basis* (GPB) with $f_1, \ldots, f_n$ the *free* invariants and $t_1, \ldots, t_d$ the *transient* invariants.

We will also be interested in the situation when $\mathbf{G}$ is an abstract group and ρ is a representation of $\mathbf{G}$ as real $n \times n$ matrices. In this case, we will denote the polynomial ring by $P(\rho)$ and the ring of invariants by $P(\rho)^{\mathbf{G}}$. We will use the notation P and $P^{\mathbf{G}}$ when $\mathbf{G}$ is assumed to be a matrix group and no reference to a representation is necessary.

The notations $P^{\mathbf{G}}$ and $P(\rho)^{\mathbf{G}}$ follow this convention: $M^{\mathbf{G}} = \{m \in M$: $\mathbf{g}m = m$ for all $\mathbf{g} \in \mathbf{G})$ whenever M is a $\mathbf{G}$-module.

It has been shown in [6] that, for any matrix group $\mathbf{G}$, a good polynomial basis always exists for $P^{\mathbf{G}}$. However, we will not need this result. For us, the major concern is to find a good polynomial basis for an arbitrary representation when bases for certain basic, related representations are known. The "basic" representations in some cases are the irreducible constituents; but, since we will also be interested in relative invariants, we will consider a slightly more general notion than irreducible constituent: twisted irreducible representations. (A representation σ is a twisted version of representation ρ if $\sigma = \mu\rho$ where μ is a linear representation and $\sigma(\mathbf{g}) = \mu(\mathbf{g})\rho(\mathbf{g})$ for all $\mathbf{g}$ in $\mathbf{G}$.)

In order to construct a good polynomial basis for reducible representations, it is useful to know not only good polynomial bases for the constituents but also the homogeneous $\mathbf{G}$-module structure of the constituent polynomial rings. Accordingly, the first sections of the paper (Sections 2–5) are concerned with the structure of P as a graded $\mathbf{G}$-algebra.

Section 2 deals with the ideal generated by $P^{\mathbf{G}}_{+}$ (the invariants with zero constant term) and its complement, following ideas of Chevalley [3], Kostant [9] and Steinberg [16]. A new bound on the degrees of a generating set of invariants is obtained and the location of new invariants in P_m, the homogeneous polynomials of degree m, is determined. (The "new" invariants in P_m are those not in the algebra generated by $P^{\mathbf{G}}_1, \ldots, P^{\mathbf{G}}_{m-1}$.) In Section 3 we recall that the classical $\mathbf{G}$-harmonic polynomials, due to Fischer [4], are a good choice for a complement to the ideal generated by $P^{\mathbf{G}}_{+}$. In Section 4 we look at how these notions for $P(\rho)$ and $P(\sigma)$ are related to those for $P(\rho \oplus \sigma)$. In Section 5 we reintroduce classical polar operators and extend the well-known results about them due to Capelli [2] and Weyl [17]. These results will give us information about the structure of the polynomial ring $P(\bigoplus^m \rho)$, the ideal of $P(\bigoplus^m \rho)$ generated by $P(\bigoplus^m \rho)^{\mathbf{G}}_{+}$, a complement of this ideal, and the new bound (first introduced in Section 2) on the degrees of the invariants.

In the second part of the paper (Sections 6 through 9) we face directly the task of finding polynomial bases for certain representations, given that bases for certain other representations are known. In Section 6 we state and prove some general results about good polynomial bases. In Section 7, when $\mathbf{H}$ is a subgroup of $\mathbf{G}$, we show how a good polynomial basis for $P^{\mathbf{H}}$ can be obtained given that a good polynomial basis for $P^{\mathbf{G}}$ is in hand. We will also show how to construct a good polynomial basis for $P(\rho \oplus \sigma)$ given bases for and additional structural information about $P(\rho)$ and $P(\sigma)$. This development follows closely the work of Kopský [8], Sloane [13], Solomon [14] and Stanley [15] on good polynomial bases.

In Section 8, we will prove some theorems about real relative invariants and make sense of the notion of good polynomial basis for real relative invariants.

Finally, in Section 9 we will bring together all the tools developed up to then and present an algorithm for constructing a good polynomial basis for relative invariants in case the corresponding representation is a sum $\bigoplus^m \mu_i \rho$ of twisted representations.

We have restricted the coefficients for our representations to be elements of the real field because this is where the applications are most numerous and the results simplest. Many parts of the paper, such as Sections 2, 3 and 6, are directly valid for other fields. Other parts are valid with minor modifications.

2. THE STRUCTURE OF THE POLYNOMIAL RING: INVARIANT IDEAL AND ITS COMPLEMENT

Let I be the ideal in P generated by $P^{\mathbf{G}}_+$, the invariant polynomials with zero constant term. If $I_m = I \cap P_m$, then it is not difficult to see that $I = \bigoplus_m I_m$ and that $I_m = \sum_{k=1}^m P^{\mathbf{G}}_k P_{m-k}$ (the latter is not necessarily a direct sum). For all k, the subspace $P^{\mathbf{G}}_k P_{m-k}$ is a **G**-submodule. Thus I_m is also a **G**-module and consequently by Maschke's Theorem it has a **G**-submodule complement Q_m in P_m:

$$P_m = I_m \oplus Q_m.$$

Let $Q = \bigoplus_m Q_m$ and call Q a *complement* to I in P. The following gives some properties of Q and indicates some relationships between it, P and $P^{\mathbf{G}}$.

THEOREM 1

(1) *The space Q is a finite dimensional vector space over $\mathbb{R}$.*
(2) *There exists an integer m so that $Q_m = \{0\}$.*
(3) *We have $P = QP^{\mathbf{G}}$. In particular, for all m, $P_m = \sum_{k=0}^m P^{\mathbf{G}}_k Q_{m-k}$ (not necessarily a direct sum).*
(4) *If $Q_m = \{0\}$, then $Q_{m+1} = \{0\}$.*
(5) *If $p \in P^{\mathbf{G}}_m$, $m \geqslant 1$, then $p \in \langle \{P^{\mathbf{G}}_j : j < m\} \rangle + Q_k Q_{m-k}$ for any $k = 1, \ldots, m-1$.*
(6) *If $Q_m = \{0\}$ for some $m > 0$, then $P^{\mathbf{G}}_1, \ldots, P^{\mathbf{G}}_m$ generate $P^{\mathbf{G}}$.*

(*Note*: If R is a ring and $S \subseteq R$, then $\langle S \rangle$ denotes the subring generated by S.)

Proof Proofs of parts (1) and (3) can be found elsewhere ([4], [14], [16]). Part (2) follows easily from (1).

(4) Suppose $Q_m = \{0\}$ and let $p \in P_{m+1}$. Then $p = \sum_i p_i q_i$ with $p_i \in P_m$ and $q_i \in P_1$. Since $Q_m = \{0\}$, $p_i \in I$. Thus also $p \in I$. Consequently, $Q_{m+1} = \{0\}$.

(5) We break the proof of (5) into five steps, (i)–(v).

(i) If $p \in P_m$, let $\tau(p) = 1/|\mathbf{G}| \sum_{\mathbf{g} \in \mathbf{G}} \mathbf{g}p$ where $|\mathbf{G}|$ = order of $\mathbf{G}$. Recall that τ is a $\mathbf{G}$-module homomorphism projecting P_m onto $P_m^{\mathbf{G}}$. Thus, if $p \in P_m^{\mathbf{G}}$, then $\tau(p) = p$. Furthermore, if $P_m = P_m^{\mathbf{G}} \oplus C$ is a $\mathbf{G}$-module direct sum and $q \in C$, then $\tau(q) = 0$. In particular, if Q is a complement to I and $q \in Q_m$, then $\tau(q) = 0$. Finally,

$$\text{if } 1 \leqslant l \leqslant m,\ u \in P_l^{\mathbf{G}},\ v \in P_{m-l},\ \text{then } \tau(uv) = u\tau(v). \qquad (*_1)$$

Indeed,

$$\tau(uv) = \frac{1}{|\mathbf{G}|} \sum_{\mathbf{g} \in \mathbf{G}} \mathbf{g}(uv) = \frac{1}{|\mathbf{G}|} \sum_{\mathbf{g} \in \mathbf{G}} \mathbf{g}(u)\mathbf{g}(v)$$
$$= \frac{1}{|\mathbf{G}|} \sum_{\mathbf{g} \in \mathbf{G}} u\mathbf{g}(v) = u \frac{1}{|\mathbf{G}|} \sum_{\mathbf{g} \in \mathbf{G}} \mathbf{g}v = u\tau(v).$$

(ii) Let k be such that $m > k \geqslant 1$. Then $P_m = P_{m-k}P_k$. Thus, from part (3) of this theorem,

$$P_m = P_k P_{m-k} = \sum_{\substack{(l,j) \neq (0,0) \\ 0 \leqslant l \leqslant k,\, 0 \leqslant j \leqslant m-k}} P_l^{\mathbf{G}} Q_{k-l} P_j^{\mathbf{G}} Q_{m-k-j} + Q_k Q_{m-k}. \quad (*_2)$$

(iii) Let $P_+^{\mathbf{G}} = \sum_{j=1}^{\infty} P_j^{\mathbf{G}}$ and $Q_+ = \sum_{j=1}^{\infty} Q_j$. From (3) an element $q \in Q_a Q_b$ ($(a, b) \neq (0, 0)$), by virtue of the fact that it is an element of P_{a+b}, is of the form

$$q = \hat{p} + \hat{r} + \hat{q}, \qquad \hat{p} \in P_+^{\mathbf{G}}, \quad \hat{r} \in P_+^{\mathbf{G}} Q_+, \quad \hat{q} \in Q_{a+b}. \qquad (*_3)$$

(iv) For the remainder of the proof of (5), fix k such that $m > k \geqslant 1$. Then from (iii) an element of the summand $P_l^{\mathbf{G}} Q_{k-l} P_j^{\mathbf{G}} Q_{m-k-j}$ ($(l, j) \neq (0, 0)$) in formula $(*_2)$ is of the form $p + r$ where $p \in P_l^{\mathbf{G}} P_j^{\mathbf{G}} P_{m-l-j}^{\mathbf{G}}$ and $r \in P_+^{\mathbf{G}} Q_+$. Thus, from $(*_2)$, an element s of P_m is of the form

$$s = p + r + q, \qquad p \in \left\langle \sum_{l=1}^{m-1} P_l^{\mathbf{G}} \right\rangle, \quad r \in P_+^{\mathbf{G}} Q_+, \quad q \in Q_k Q_{m-k}. \quad (*_4)$$

Combining $(*_3)$ and $(*_4)$, we get

$$s = p + r + q = p + r + \hat{p} + \hat{r} + \hat{q}, \qquad (*_5)$$

$$p \in \left\langle \sum_{l=1}^{m-1} P_l^{\mathbf{G}} \right\rangle, \quad \hat{p} \in P_m^{\mathbf{G}} \cap Q_k Q_{m-k}, \quad r + \hat{r} \in P_+^{\mathbf{G}} Q_+, \quad \hat{q} \in Q_m.$$

(v) Assume now that $s \in p_m^{\mathbf{G}}$. To prove (3) we want to show that s can be written as $s = s_1 + s_2$ where $s_1 \in \langle \sum_{l=1}^{m-1} P_l^{\mathbf{G}} \rangle$ and $s_2 \in Q_k Q_{m-k}$ for k (fixed) such that $m > k \geqslant 1$. From $(*_5)$ we have that $s = (p + \hat{p}) + (r + \hat{r}) + \hat{q}$. Since $s \in P_m^{\mathbf{G}}$, we know that $\tau(s) = s$. Using (i), we get $\tau(\hat{q}) = 0$. Using formula $(*_1)$, we get $\tau(r + \hat{r}) = 0$. Thus

$$\begin{aligned} s = (p + \hat{p}) + (r + \hat{r}) + \hat{q} &= \tau((p + \hat{p}) + (r + \hat{r}) + (\hat{q})) \\ &= \tau(p + \hat{p}) + \tau(r + \hat{r}) + \tau(\hat{q}) \\ &= \tau(p + \hat{p}) + 0 + 0 \\ &= p + \hat{p}. \end{aligned}$$

This completes the proof of (5).

(6) This follows immediately from (5). ■

Denote by $q_{\mathbf{G}}$ the largest m such that $Q_m \neq \{0\}$ and by $i_{\mathbf{G}}$ the smallest m such that $P_1^{\mathbf{G}}, \ldots, P_m^{\mathbf{G}}$ generate $P^{\mathbf{G}}$. Theorem 1 says $i_{\mathbf{G}} \leqslant q_{\mathbf{G}} + 1$. Noether's theorem [11] says $i_{\mathbf{G}} \leqslant |\mathbf{G}|$.

3. A NATURAL COMPLEMENT: G-HARMONIC POLYNOMIALS

We may assume without loss of generality that $\mathbf{G} \subseteq \mathbf{O}(n)$ and that the X_i's form an orthonormal basis of P_1. Define an inner product on P_m by assuming that the polynomial basis of monomials is orthogonal and that the inner product of $X_{i_1}^{n_1} \cdots X_{i_k}^{n_k}$ $(1 \leqslant i_1 < \cdots < i_k \leqslant n)$ with itself is $n_1! \cdots n_k!$. Assuming that P_m is orthogonal to P_k whenever m and k are distinct, we get an inner product $p \cdot q$ on P.

PROPOSITION 1 *$p \cdot q$ is* **G**-*invariant.*

Thus a natural complement to I_m is $(I_m)^{\perp}$, the orthogonal complement to I_m in P_m. An alternative description of this complement is as follows:

Let $p \in P_m$ with $p = \sum \alpha_{i_1 \cdots i_m} X_{i_1} \cdots X_{i_m}$, $\alpha_{i_1 \cdots i_m} \in \mathbb{R}$. Define $D_p =$

$\sum \alpha_{i_1\cdots i_m}(\partial/\partial X_{i_1})\cdots(\partial/\partial X_{i_m})$. It is not difficult to see that $p\cdot q = D_p q|_{X=0}$ where $X = (X_1,\cdots,X_n)$.

If p is not homogeneous, define D_p in the obvious manner. Then each D_p operates on P in the usual way. For all $m \geqslant 0$, let

$$H_m = \{q \in P_m\colon D_p(q) = 0 \text{ for all } p \in P^{\mathbf{G}}_+\}$$

and set $H = \bigoplus_m H_m$. It is clear that

$$H = \{q \in P\colon D_p(q) = 0 \text{ for all } p \in P^{\mathbf{G}}_+\}.$$

We call H the **G**-*harmonic polynomials.*

PROPOSITION 2 (1) $H_m = (I_m)^\perp$. *Thus H is a complement to I.* (2) *If B is a generating set for $P^{\mathbf{G}}$, then $H_m = \{q \in P_m\colon D_p(q)$ for all $p \in B\}$.*

Propositions 1 and 2 are well known ([4], [5], [9] and [16]). For completeness we prove them here.

LEMMA 1 *Let $\mathbf{g} \in \mathbf{GL}(n)$ and $\mathbf{g}^{-1}X_i = \sum_j \tilde{a}_{ji}X_j$. Then* (a) *if $p \in P$, $\mathbf{g}(\partial p/\partial X_k) = \sum_l \tilde{a}_{kl}(\partial \mathbf{g}p/\partial X_l)$ for all k and* (b) *if $p, q \in P$ and $\mathbf{g} \in \mathbf{O}(n)$, then $\mathbf{g}D_p q = D_{\mathbf{g}p}\mathbf{g}q$.*

Proof (a) Let $\mathbf{g}X_i = \sum a_{ji}X_j = Z_i$, $X = (X_1, \ldots, X_n)$, and $Z = (Z_1, \ldots, Z_n)$. Then $\mathbf{g}p = \mathbf{g}p(X_1, \ldots, X_n) = p(Z_1, \ldots, Z_n)$. By the chain rule, we have

$$\frac{\partial \mathbf{g}p(X)}{\partial X_j} = \frac{\partial p(Z)}{\partial X_j} = \sum_i \frac{\partial Z_i}{\partial X_j}\frac{\partial p(Z)}{\partial Z_i} = \sum a_{ji}\frac{\partial p(Z)}{\partial Z_i}. \qquad (*)$$

It is easy to check, on monomials, that $[\partial p(Z)/\partial Z_i] = \mathbf{g}[\partial p(X)/\partial X_i]$. From this and $(*)$ we have

$$\mathbf{g}\frac{\partial p(X)}{\partial X_k} = \sum_l \tilde{a}_{kl}\frac{\partial \mathbf{g}p(X)}{\partial X_l}.$$

(b) By (a), this is true for $p = X_i$, any i and any q. Thus if p is the monomial $X_{i_1}\cdots X_{i_k}$, then

$$\begin{aligned}\mathbf{g}D_{X_{i_1}\cdots X_{i_k}}q &= \mathbf{g}\frac{\partial}{\partial X_{i_1}}(D_{X_{i_2}\cdots X_{i_k}}q)\\ &= \frac{\partial}{\partial \mathbf{g}X_{i_1}}\mathbf{g}(D_{X_{i_2}\cdots X_{i_k}}q)\end{aligned}$$

$$= \frac{\partial}{\partial \mathbf{g} X_{i_1}} \cdots \frac{\partial}{\partial \mathbf{g} X_{i_k}} \mathbf{g} q \quad \text{(by induction)}$$

$$= D_{(\mathbf{g}X_{i_1})\cdots(\mathbf{g}X_{i_k})} \mathbf{g} q$$

$$= D_{\mathbf{g}p} \mathbf{g} q.$$

It is clear that this result extends to all polynomials by linearity. ■

Proof of Proposition 1 $\mathbf{g}p \cdot \mathbf{g}q = D_{\mathbf{g}p}\mathbf{g}q|_{X=0} = \mathbf{g}D_p q|_{X=0} =$ constant term of $\mathbf{g}D_p q =$ constant term of $D_p q = D_p q|_{X=0} = p \cdot q$. ■

Proof of Proposition 2 Let $J_m = P_m \cap (I_m)^\perp$. We want to show $H_m = J_m$. Let $h \in H_m$ and $p \in I_m$. Then $p = \sum s_i t_i$ where $s_i \in P_+^{\mathbf{G}}$, $t_i \in P$. Since $D_{s_i} h = 0$ for all i, we must have $D_p h = \sum_i D_{t_i} D_{s_i} h = 0$. Consequently, $p \cdot h = 0$. Thus $H_m \subseteq J_m$.

To show the inclusion in the other direction, we first observe that $(st) \cdot p = t \cdot D_s p$ for all $s, t, p \in P$. Now let $p \in J_m$. Then $(st) \cdot p = 0$ for all $s \in P_k^{\mathbf{G}}$, $t \in P_{m-k}$, $0 < k \leqslant m$. But $st \cdot p = t \cdot D_s p = 0$. Since the latter equality is true for all $t \in P_{m-k}$, we have $D_s p = 0$. Thus $D_s p = 0$ for all $s \in P_k^{\mathbf{G}}$ with $0 < k \leqslant m$. Since $D_r p = 0$ for any $r \in P_l$, $l > m$, we have that $D_r p = 0$ for all $r \in P_+^{\mathbf{G}}$. Thus $p \in H_m$.

Since $P_m = I_m \oplus J_m$ as vector spaces, and since $H_m = J_m$, we have $P_m = I_m \oplus H_m$ as vector spaces. Since the inner product on P_m is $\mathbf{G}$-invariant by Proposition 1 we have that $P_m = I_m \oplus H_m$ as $\mathbf{G}$-modules. Hence $H = \bigoplus_m H_m$ is a complement. ■

4. THE STRUCTURE OF THE POLYNOMIAL RING: REDUCIBLE REPRESENTATIONS

For a fixed orthogonal representation ρ of $\mathbf{G}$, denote by $P(\rho)$, $P(\rho)^{\mathbf{G}}$, $I(\rho)$, $H(\rho)$, $q_{\mathbf{G}}(\rho)$ and $i_{\mathbf{G}}(\rho)$ the ring of polynomials, invariants, invariant ideal, harmonics and bounds (respectively) associated with the matrix group $\rho(\mathbf{G})$.

Let ρ and σ be two representations of $\mathbf{G}$ and $\rho \oplus \sigma$ their direct sum. We would like to describe the invariant and the harmonic polynomials associated with the matrix group $\rho \oplus \sigma(\mathbf{G})$ given that the same are known for $\rho(\mathbf{G})$ and $\sigma(\mathbf{G})$.

The following theorem, similar to Theorem 1, is a first step.

THEOREM 2

(1) $P_m(\rho \oplus \sigma) = \bigoplus_{k=0}^{m} P_k(\rho)P_{n-k}(\sigma)$;

(2) $H_m(\rho \oplus \sigma) \subseteq \bigoplus_{k=0}^{m} H_k(\rho)H_{m-k}(\sigma)$;

(3) *for all* $m \geqslant 1$,

$$P_m(\rho \oplus \sigma)^{\mathbf{G}} \subseteq \langle P_j(\rho)^{\mathbf{G}}, P_j(\sigma)^{\mathbf{G}}, P_{j-1}(\rho \oplus \sigma)^{\mathbf{G}}: 1 \leqslant j \leqslant m\rangle + \sum_{k=1}^{m} H_k(\rho)H_{m-k}(\sigma);$$

(4) $q_{\mathbf{G}}(\rho) + q_{\mathbf{G}}(\sigma) \geqslant q_{\mathbf{G}}(\rho \oplus \sigma) \geqslant \max\{q_{\mathbf{G}}(\rho), q_{\mathbf{G}}(\sigma)\}$;

(5) $q_{\mathbf{G}}(\rho) + q_{\mathbf{G}}(\sigma) + 1 \geqslant i_{\mathbf{G}}(\rho \oplus \sigma) \geqslant \max\{i_{\mathbf{G}}(\rho), i_{\mathbf{G}}(\sigma)\}$.

Proof (1) follows from the fact that $P(\rho \oplus \sigma) = P(\rho)P(\sigma)$ and that $P(\rho) = \bigoplus_k P_k(\rho)$, $P(\sigma) = \bigoplus_k P_k(\sigma)$.

(2) We know that $P_k(\sigma) = I_k(\sigma) \oplus H_k(\sigma)$ and $P_l(\rho) = I_l(\rho) \oplus H_l(\rho)$. Let

$$L = \bigoplus_{k+l=m} H_k(\sigma)H_l(\rho)$$

and

$$K = \bigoplus_{\substack{k+l=m \\ k \neq 0, l \neq 0}} [I_k(\sigma)I_l(\rho) \oplus I_k(\sigma)H_l(\rho) \oplus H_k(\sigma)I_l(\rho)] \oplus I_m(\sigma) \oplus I_m(\rho).$$

Then, from (1) we have

$$P_m(\sigma \oplus \rho) = \bigoplus_{k+l=m} P_k(\sigma)P_l(\rho) = K \oplus L.$$

We claim that $L \perp K$. Indeed, L is spanned by elements of the form pq where $p \in H(\sigma)$, $q \in H(\rho)$. Also, K is spanned by elements of the form st where either $s \in P_+^{\mathbf{G}}(\sigma)$ or $s \in P_+^{\mathbf{G}}(\rho)$ and $t \in P(\rho \oplus \sigma)$, with suitable restrictions on the degrees of s and t. Assume $s \in P_+^{\mathbf{G}}(\sigma)$. (The case $s \in P_+^{\mathbf{G}}(\rho)$ is similar). Then $st \cdot pq = t \cdot D_s(pq) = t \cdot (D_s p)q$ (since the variables of s and q are disjoint) $= 0$ (since p is harmonic in $P(\sigma)$). This proves the claim. Since $H_m(\sigma \oplus \rho)$ must also be orthogonal to K, it must be that $H_m(\sigma \oplus \rho) \subseteq L$. This completes the proof of (2).

The proof of (3) is entirely analogous to the proof of Theorem 1 part (5). (4) and (5) follow easily from (1), (2) and (3) above. ■

5. POLARIZATION: THE STRUCTURE OF THE POLYNOMIAL RING FOR $\bigoplus^m \rho$

Among all reducible representation of $\mathbf{G}$, we are particularly interested in those of the form $\rho \oplus \rho \oplus \cdots \oplus \rho$. For such representations, the very crude estimates for $q_{\mathbf{G}}$ and $i_{\mathbf{G}}$ implied by Theorem 2 can be improved considerably using a classical tool known as the polar operator. We shall describe how to do this below and also extend the classical results by showing how polar operators behave on harmonic polynomials.

Let $\rho : \mathbf{G} \to \mathbf{O}(n)$ be a representation of $\mathbf{G}$ with corresponding polynomial basis $X_1, \ldots, X_n$. For $\mathbf{h} \in \rho(\mathbf{G})$ suppose $\mathbf{h}X_i = \sum_j a_{ji} X_j$. For $l = 1, \ldots, m$ let $X_1^{(l)}, \ldots, X_n^{(l)}$ be polynomial variables with $\mathbf{h}X_i^{(l)} = \sum_j a_{ji} X_j^{(l)}$, i.e. $\rho(\mathbf{G})$ acts on $X_1^{(l)}, \ldots, X_n^{(l)}$ just as it acts on $X_1, \ldots, X_n$. Furthermore, assume that the $X_i^{(l)}$, $i = 1, \ldots, n$, $l = 1, \ldots, m$ are algebraically independent, forming a polynomial basis associated with the representation

$$\bigoplus^m \rho = \underbrace{\rho \oplus \cdots \oplus \rho}_{m\text{-copies}}.$$

For $i, j = 1, \ldots, m$ we define an operator D_{ij} on $P(\bigoplus^m \rho)$ by

$$D_{ij} p = \sum_k X_k^{(i)} \frac{\partial}{\partial X_k^{(j)}} p,$$

called *polarization of p with respect to* $(X_1^{(i)}, \ldots, X_n^{(i)})$ *at* $(X_1^{(j)}, \ldots, X_n^{(j)})$. Some important properties of such an operator are given by the following.

PROPOSITION 3 *If $\mathbf{h} \in \rho(\mathbf{G})$, then $\mathbf{h}(D_{ij}p) = D_{ij}(\mathbf{h}p)$. Thus a polar operator is a $\mathbf{G}$-module homomorphism. In other words, if $U \subseteq P_r(\bigoplus^m \rho)$ is a simple $\mathbf{G}$-submodule, then either $D_{ij}(U)$ and U are isomorphic as $\mathbf{G}$ modules or $D_{ij}(U) = \{0\}$. In particular, if p is invariant so is $D_{ij}p$.*

Proof Let $\mathbf{g}X_k^{(i)} = \sum_l a_{lk} X_l^{(i)}$ and $\mathbf{g}^{-1}X_k^{(i)} = \sum_l \tilde{a}_{lk} X_l^{(i)}$. Then by Lemma 1(a)

$$\mathbf{g}\,\frac{\partial p}{\partial X_k^{(i)}} = \sum_l \tilde{a}_{kl}\,\frac{\partial \mathbf{g}p}{\partial X_l^{(i)}}$$

Then

$$\begin{aligned}\mathbf{g}D_{ij}p &= \sum_k (\mathbf{g}X_k^{(i)})\left(\mathbf{g}\,\frac{\partial p}{\partial X_k^{(j)}}\right) = \sum_{k,l,h} a_{lk} X_l^{(i)} \tilde{a}_{kh}\,\frac{\partial \mathbf{g}p}{\partial X_h^{(j)}} \\ &= \sum_{l,h}\left(\sum_k a_{lk}\tilde{a}_{kh}\right) X_l^{(i)}\,\frac{\partial \mathbf{g}p}{\partial X_h^{(j)}} = \sum_{h,l} \delta_{hl} X_l^{(i)}\,\frac{\partial \mathbf{g}p}{\partial X_h^{(j)}} \\ &= \sum_l X_l^{(i)}\,\frac{\partial \mathbf{g}p}{\partial X_l^{(j)}} = D_{ij}\mathbf{g}p(X). \qquad \blacksquare\end{aligned}$$

For a subset $S \subseteq P(\bigoplus^m \rho)$, we denote by $\mathrm{Pol}(S)$ the vector space span of those $q \in P(\bigoplus^m \rho)$ for which there exists $p \in S$ and integers $1 \leqslant i_1, \ldots, i_k, j_1, \ldots, j_k \leqslant m$ so that $q = D_{i_1 j_1} \cdots D_{i_k j_k} p$. We say that an element of $\mathrm{Pol}(S)$ is obtained from S by the *polar process*. The principal facts about this (old and new) are contained in the following.

THEOREM 3 *Let* $m \geqslant n = \dim \rho$. Then

(1) $P\left(\overset{m}{\bigoplus} \rho\right) = \mathrm{Pol}\left(P\left(\overset{n}{\bigoplus} \rho\right)\right)$,

(2) $P\left(\overset{m}{\bigoplus} \rho\right)^{\mathbf{G}} = \mathrm{Pol}\left(P\left(\overset{n}{\bigoplus} \rho\right)^{\mathbf{G}}\right)$,

(3) $i_{\mathbf{G}}\left(\overset{m}{\bigoplus} \rho\right) = i_{\mathbf{G}}\left(\overset{n}{\bigoplus} \rho\right)$,

(4) $I\left(\overset{m}{\bigoplus} \rho\right) = \mathrm{Pol}\left(I\left(\overset{n}{\bigoplus} \rho\right)\right)$,

(5) $H\left(\overset{m}{\bigoplus} \rho\right) = \mathrm{Pol}\left(H\left(\overset{n}{\bigoplus} \rho\right)\right)$,

(6) $q_{\mathbf{G}}\left(\overset{m}{\bigoplus} \rho\right) = q_{\mathbf{G}}\left(\overset{n}{\bigoplus} \rho\right)$.

Before proving the theorem, we first state and prove a lemma.

LEMMA 2 (a) *Let $s \leqslant m$. Assume the natural inclusion of $P(\bigoplus^s \rho)$ in $P(\bigoplus^m \rho)$. Then $H(\bigoplus^s \rho) \subseteq H(\bigoplus^m \rho)$.* (b) *Let $i, j \leqslant m$ and $p \in H(\bigoplus^m \rho)$. Then $D_{ji} p \in H(\bigoplus^m \rho)$.*

Proof (a) Let $h \in H(\bigoplus^s \rho)$ and $p \in P_+^{\mathbf{G}}(\bigoplus^m \rho)$. Without loss of generality we may assume $p \in P_{n_1}(\rho_1) \cdots P_{n_m}(\rho_m)$ where $\rho_1, \ldots, \rho_m$ denote the m copies of ρ. We know $P(\bigoplus^s \rho) = P(\rho_1) \cdots P(\rho_s)$. Thus, if one of $n_{s+1}, \ldots, n_m$ is not zero, then $D_p h = 0$. If $n_{s+1} = \cdots = n_m = 0$, then $p \in P_+^{\mathbf{G}}(\bigoplus^s \rho)$ and $D_p h = 0$.

(b) Let $q \in P_+(\bigoplus^m \rho)^{\mathbf{G}}$. Therefore $p \in H(\bigoplus^m \rho)$ implies $D_q p = 0$. We want to show that $D_q(D_{ji} p) = 0$ also.

Write $q = \sum m_l$ where m_l is a monomial (with some real coefficient) and, for each k, $m_l = (X_k^{(j)})^{h_{kl}} d_{kl}$ where h_{kl} is a non-negative integer and d_{kl} is a monomial not divisible by $X_k^{(j)}$. Thus

$$\begin{aligned}
D_q(D_{ji} p) &= \sum_k D_q X_k^{(j)} \frac{\partial}{\partial X_k^{(i)}} p \\
&= \sum_k \left(\sum_l D_{m_l} \right) X_k^{(j)} \frac{\partial}{\partial X_k^{(i)}} p \\
&= \sum_{k,l} D_{d_{kl}} \left(\frac{\partial}{\partial X_k^{(j)}} \right)^{h_{kl}} X_k^{(j)} \frac{\partial}{\partial X_k^{(i)}} p \\
&= \sum_{k,l} h_{kl} D_{d_{kl}} \left(\frac{\partial}{\partial X_k^{(j)}} \right)^{h_{kl}-1} \frac{\partial}{\partial X_k^{(i)}} p \\
&\quad + \sum_{k,l} D_{d_{kl}} X_k^{(j)} \left(\frac{\partial}{\partial X_k^{(j)}} \right)^{h_{kl}} \frac{\partial}{\partial X_k^{(i)}} p \\
&= D_{D_{ij}q} p + D_{ji}(D_q p).
\end{aligned}$$

Since $q \in P(\bigoplus^m \rho)_+^{\mathbf{G}}$ and $p \in H(\bigoplus^m \rho)$, we have that $D_q p = 0$ and hence that the second term, $D_{ji}(D_q p)$, is 0. Also, since $q \in P(\bigoplus^m \rho)_+^{\mathbf{G}}$, we have $D_{ij} q \in P(\bigoplus^m \rho)_+^{\mathbf{G}}$ by Proposition 3. Again, by the definition of harmonic, we have $D_{D_{ij}q} p = 0$. This proves Lemma 2. ■

Proof of Theorem Parts (1), (2) and (3) of the theorem are proved in Weyl [17, p. 43f].

To prove (4) and (5), we first note that a polar operator D_{ij} is a derivation, i.e.

$$D_{ij}(pq) = (D_{ij}p)q + p(D_{ij}q). \qquad (*)$$

In particular, if $p \in P(\bigoplus^m \rho)^{\mathbf{G}}_+$ then by Proposition 3 $D_{ij}p \in P(\bigoplus^m \rho)^{\mathbf{G}}_+$ and so also $D_{ij}(pq) \in I(\bigoplus^m \rho)$ from equation $(*)$. Thus $\mathrm{Pol}(I(\bigoplus^n \rho)) \subseteq I(\bigoplus^m \rho)$. At the same time Lemma 2 implies $\mathrm{Pol}(H(\bigoplus^n \rho)) \subseteq H(\bigoplus^m \rho)$.

To show both of these inclusions in the other direction, let $h \in H(\bigoplus^m \rho)$. Then by part (1) of this theorem and by what we have just shown, $h = q + \hat{h}$, where $q \in \mathrm{Pol}(I(\bigoplus^n \rho))$ and $\hat{h} \in \mathrm{Pol}(H(\bigoplus^n \rho))$. Hence we must have $q = 0$ and consequently $H(\bigoplus^m \rho) = \mathrm{Pol}(H(\bigoplus^n \rho))$. Similarly, $I(\bigoplus^m \rho) = \mathrm{Pol}(I(\bigoplus^n \rho))$. This proves (4) and (5) from which (6) follows immediately. ■

Parts (1), (2) and (3) of the theorem reduce the problem of finding a generating set for $P(\bigoplus^m \rho)^{\mathbf{G}}$ to that of finding one for $P(\bigoplus^n \rho)^{\mathbf{G}}$ and using the polar process. The bound $i_{\mathbf{G}}(\bigoplus^m \rho)$ is equal to $i_{\mathbf{G}}(\bigoplus^n \rho)$.

Parts (4), (5) and (6) of the theorem are new and determine $q_{\mathbf{G}}(\bigoplus^m \rho)$ once $q_{\mathbf{G}}(\bigoplus^n \rho)$ is known. Thus if σ and ρ are two inequivalent irreducible representations of $\mathbf{G}$ of degrees K and n respectively, then Theorems 2 and 3 say that

$$i_{\mathbf{G}}\left(\left(\bigoplus^{l} \sigma\right) \oplus \left(\bigoplus^{m} \rho\right)\right) \leqslant q_{\mathbf{G}}\left(\bigoplus^{k} \sigma\right) + q_{\mathbf{G}}\left(\bigoplus^{n} \rho\right) + 1$$

when $l \geqslant k$ and $m \geqslant n$.

Theorem 3 part (5) together with Proposition 3 also says something about the $\mathbf{G}$-module structure of $H_k(\bigoplus^m \rho)$ given that it is known for $H_k(\bigoplus^n \rho)$. For example, if M is a simple $\mathbf{G}$-submodule of $H_k(\bigoplus^n \rho)$ and D is a product of polarizations, then $D(M)$ is $\{0\}$ or is a $\mathbf{G}$-submodule of $H_k(\bigoplus^m \rho)$ isomorphic to M. Knowledge of the $\mathbf{G}$-module structure of $H_k(\bigoplus^m \rho)$ is important for finding invariants. Indeed, from Theorem 2 part (3), the interesting invariants in $P_l((\bigoplus^l \sigma) \oplus (\bigoplus^m \rho))$ are found in subspaces $H_i(\bigoplus^l \sigma)H_{l-i}(\bigoplus^m \rho)$ for $i = 1, \ldots, l-1$. (See also Theorem 8.) Such invariants can be gotten from invariants in $H_i(\bigoplus^k \sigma)H_{l-i}(\bigoplus^n \rho)$ by a kind of "mixed" polarization described in the paragraphs that follow below.

Let C and D be the harmonic polynomials for $\bigoplus^l \sigma$ and $\bigoplus^m \rho$ respectively and let C' and D' be the harmonics for $\bigoplus^k \sigma$ and $\bigoplus^n \rho$ respectively. From Theorem 3, $C = \mathrm{Pol}(C')$ and $D = \mathrm{Pol}(D')$. In particular, simple summands for C (respectively D) can be chosen to be of the form $\partial(A)$ (respectively $\partial'(B)$) where A (respectively B) is an irreducible summand of C' (respectively D') and ∂ (respectively ∂') is an appropriate product of polarizations. Thus a typical basis element for $(CD)^{\mathbf{G}}$ can be chosen to be contained in a $\partial(A)\,\partial'(B)^{\mathbf{G}}$. We know $(\partial(A)\,\partial'(B))^{\mathbf{G}} \neq \{0\}$ iff $\partial(A) \neq \{0\} \neq \partial'(B)$ and $\partial(A)$ and $\partial'(B)$ are isomorphic as **G**-modules.

By Proposition 3, we then know that $(AB)^{\mathbf{G}} \neq \{0\}$ and hence that A and B are isomorphic **G**-modules. Furthermore $(AB)^{\mathbf{G}}$ is spanned by a single invariant p, in case A is absolutely irreducible, and by a linearly independent pair of invariants p, q, in case A is real irreducible but not absolutely irreducible. Since also $\partial\partial' = \partial'\partial$, we have that $(\partial A\ \partial' B)^{\mathbf{G}}$ is spanned by $\partial\partial'(p)$, in the first case, and by the pair $\partial\partial'(p)$, $\partial\partial'(q)$, in the second.

We summarize all of this in the following proposition.

PROPOSITION 4 *Let μ, ν be two real representations of* **G**. *Let $M = (H(\mu)H(\nu))^{\mathbf{G}}$.* (1) *Then M is spanned by all elements of sets of the form $(AB)^{\mathbf{G}}$, where A is an irreducible subspace of $H(\mu)$, B is an irreducible subspace of $H(\nu)$ and A and B are isomorphic* **G**-*modules.* (2) *Moreover, if $\mu = \bigoplus^m \rho$ and $\nu = \bigoplus^l \sigma$ where $m \geqslant \dim \rho = n$ and $l \geqslant \dim \sigma = k$, then M is spanned by polynomials of the form $\partial\partial'(p)$ where p is in the spanning set for $(H(\bigoplus^n \rho)H(\bigoplus^k \sigma))^{\mathbf{G}}$ as described in* (1), *∂ is a product of polar operators from $P(\bigoplus^n \rho)$ to $P(\bigoplus^m \rho)$ and ∂' is a product of polar operators from $P(\bigoplus^k \sigma)$ to $P(\bigoplus^l \sigma)$.*

A classical improvement of Theorem 3 for a special case is the following.

PROPOSITION 5 *Let $\rho\colon \mathbf{G} \to \mathbf{O}(n)$ be a representation of* **G**. *Assuming the notation of the paragraph preceding Theorem 3, we let $\alpha = \det(X_j^{(i)})_{1 \leqslant i,j \leqslant n}$ (α is then an element of $P(\bigoplus^n \rho)$) and let B be a generating set for the invariants $P(\bigoplus^{n-1} \rho)^{\mathbf{G}}$. If $\alpha \in P(\bigoplus^n \rho)^{\mathbf{G}}$, then for $m \geqslant n-1$ $P(\bigoplus^m \rho)^{\mathbf{G}}$ is the linear span of the image of the set $B \cup \{\alpha\}$ under polar operators of the form $D_{i_1 j_1} \cdots D_{i_k j_k}$ ($1 \leqslant i_1, \ldots, i_k \leqslant m$; $1 \leqslant j_1, \ldots, j_k \leqslant n$).*

This proposition is in Weyl [17, p. 44].

Since the α of Proposition 5 is an invariant iff $\rho(\mathbf{G}) \subseteq \mathbf{SO}(n)$, the theorem is a good improvement on Theorem 3 in case $\rho(\mathbf{G})$ is a rotation group. Using Proposition 5 we also have the following.

THEOREM 4 *Assume the notation of Proposition 5. Suppose $\rho(\mathbf{G})$ is a* *rotation group. Then* $H(\bigoplus^m \rho) = \mathrm{Pol}(H(\bigoplus^{n-1} \rho))$. *In particular* $q_{\mathbf{G}}(\bigoplus^m) = q_{\mathbf{G}}(\bigoplus^{n-1} \rho)$.

Proof By Theorem 3, we have $H_l(\bigoplus^m \rho) = \mathrm{Pol}(H_l(\bigoplus^n \rho))$. By Lemma 2, we have $\mathrm{Pol}(H_l(\bigoplus^{n-1} \rho)) \subseteq H_l(\bigoplus^m \rho)$. We will prove the theorem by showing $H_l(\bigoplus^n \rho) \subseteq \mathrm{Pol}(H_l(\bigoplus^{n-1} \rho))$.

By the Capelli identities used by Weyl [17, p. 43] to prove Proposition 5, we have

$$P_l\left(\bigoplus^n \rho\right) \subseteq \mathrm{Pol}\left(P_l\left(\bigoplus^{n-1} \rho\right)\right) + \alpha P_{l-n}\left(\bigoplus^n \rho\right).$$

(If $l < n$, then the right-most term is assumed to be zero.) Thus

$$H_l\left(\bigoplus^n \rho\right) \subseteq \mathrm{Pol}\left(H_l\left(\bigoplus^{n-1} \rho\right)\right) + \mathrm{Pol}\left(I_l\left(\bigoplus^{n-1} \rho\right)\right) + \alpha P_{l-n}\left(\bigoplus^n \rho\right).$$

But, since $\alpha \in P_n(\bigoplus^n \rho)^{\mathbf{G}}$ and $\mathrm{Pol}(I_l(\bigoplus^{n-1} \rho)) \subseteq I_l(\bigoplus^m \rho)$ by Theorem 3, we have $\mathrm{Pol}(I_l(\bigoplus^{n-1} \rho)) + \alpha P_{l-n}(\bigoplus^n \rho) \subseteq I_l(\bigoplus^m \rho)$. Hence $H_l(\bigoplus^n \rho) \subseteq \mathrm{Pol}(H_l(\bigoplus^{n-1} \rho))$. This proves Theorem 4. ■

6. GENERAL RESULTS ON GOOD POLYNOMIAL BASES

Let $f_1, \ldots, f_n$ be the free invariants of a good polynomial basis for $P^{\mathbf{G}}$. Then transient invariants for the basis can be found as follows. Let J be the ideal in P generated by the free invariants. Let $J_m = P_m \cap J$ so that, because the f_i's are homogeneous, $J = \bigoplus_m J_m$. Certainly, J_m is a **G**-submodule so that there exists a **G**-submodule C_m of P_m with $P_m = J_m \oplus C_m$. Let $C = \bigoplus_m C_m$ and call C a *complement relative to* $f_1, \ldots, f_n$. (This construction is identical to the construction of the complement Q to the ideal $I = P^{\mathbf{G}}_+ P$ in Section 2.) By analogy with Theorems 1 (parts 1–4) and 2 (part 3), we have the following.

THEOREM 5 *Let* $F = \mathbb{R}[f_1, \ldots, f_n]$ *and* $C^{\mathbf{G}} = \{p \in C: \mathbf{g}p = p,$ *all* $\mathbf{g} \in \mathbf{G}\}$. *The spaces* C *and* $C^{\mathbf{G}}$ *have the following properties*:

(1) $\dim C < \infty$.

(2) $P = FC$ *and, in particular,*

$$P_m = \bigoplus_{k=0}^{m} F_k C_{m-k}, \qquad \text{where} \quad F_k = P_k \cap F.$$

(3) $P_m^{\mathbf{G}} = \bigoplus_{k=0}^{m} F_k C_{m-k}^{\mathbf{G}}$.

(4) *Any homogeneous basis for $C^{\mathbf{G}}$ together with the free invariants $f_1, \ldots, f_n$ form a good polynomial basis for $P^{\mathbf{G}}$. Conversely, the transients in a good polynomial basis, with $f_1, \ldots, f_n$ as free invariants, form a basis for $C^{\mathbf{G}}$ for some complement C.*

Proof This theorem follows almost immediately from the following restatement:

The homogeneous polynomials $f_1, \ldots, f_n$ are the free invariants for a good polynomial basis iff, when $F = \mathbb{R}[f_1, \ldots, f_n]$, there exists $C = \bigoplus_{i=1}^{k} C_i$ (a $\mathbf{G}$-submodule of P with $C_i \subseteq P_i$) so that

$$P = FC \cong F \otimes C$$

i.e. $P_m = \bigoplus_k F_k C_{m-k}$. Furthermore, C is a complement to F.

Proof of ($\Rightarrow$) is found in Solomon [14].

($\Leftarrow$) Let $\eta_1, \ldots, \eta_l$ be a homogeneous basis for $C^{\mathbf{G}}$ and $\eta_0 = 1$. Then $P^{\mathbf{G}} = \sum_{i=0}^{l} \eta_i F$ is a direct sum. This is the definition of a GPB with $f_1, \ldots, f_n$ (free) and $\eta_1, \ldots, \eta_l$ (transients). ■

According to Theorem 5, if $f_1, \ldots, f_n$ are the free invariants and $t_1, \ldots, t_k$ some of the transient invariants of a good polynomial basis, then there are many possible choices for polynomials $s_1, \ldots, s_h$ so that $f_1, \ldots, f_n$ (free), $t_1, \ldots, t_k, s_1, \ldots, s_h$ (transient) is also a good polynomial basis. The latter is called a *completion* of the former. If $f_1, \ldots, f_n$ are the free invariants of a good polynomial basis and $t_1, \ldots, t_k$ are the transients of degree $\leqslant m$ for some completion of $f_1, \ldots, f_n$, then we say that $f_1, \ldots, f_n, t_1, \ldots, t_k$ is a *partial completion of* $f_1, \ldots, f_n$ *up to degree* m. Here is a method for obtaining a completion of $f_1, \ldots, f_n$ (free) through partial completions.

THEOREM 6 *Let S be a set of polynomials spanning $P_{m+1}^{\mathbf{G}}$ and let $f_1, \ldots, f_n, t_1, \ldots, t_k$ be a partial completion of $f_1, \ldots, f_n$ up to degree m. Let $d_i = \deg(t_i)$ and $F_k = P_k \cap \mathbb{R}[f_1, \ldots, f_n]$. Select a maximal linearly independent subset B of S whose span $[B]$ has the property $[B] \cap \bigoplus_{d_i+k=m+1} t_i F_k = \{0\}$. Let $B = \{s_1, \ldots, s_h\}$. Then $f_1, \ldots, f_n$, $t_1, \ldots, t_k$, $s_1, \ldots, s_h$ is a partial completion of $f_1, \ldots, f_n$ up to degree $m + 1$.*

Proof Let J be the ideal generated by $f_1, \ldots, f_n$ and let $J_k = J \cap P_k$. The invariant polynomials we seek will be a basis for $C^{\mathbf{G}}_{m+1}$ for some complement C relative to $f_1, \ldots, f_n$. For $i = 1, \ldots, m$, let C_i be such that $P_i = C_i \oplus J_i$ and $t_1, \ldots, t_k$ is a homogeneous basis for $(C_1 \oplus \cdots \oplus C_m)^{\mathbf{G}}$. ($C_1 \oplus \cdots \oplus C_m$ is a "partial" complement relative to $f_1, \ldots, f_n$.) If C_{m+1} is such that $P_{m+1} = J_{m+1} \oplus C_{m+1}$, then $f_1, \ldots, f_n$ algebraically independent implies that $P_{m+1} = \sum_{k=0}^{m+1} F_k C_{m+1-k}$ is a direct sum. Thus also $J_{m+1} = \bigoplus_{k=1}^{m+1} F_k C_{m+1-k}$ and hence

$$J^{\mathbf{G}}_{m+1} = \bigoplus_{k=1}^{m+1} F_k C^{\mathbf{G}}_{m+1-k} = \bigoplus_{\substack{k \geqslant 1 \\ d_i + k = m+1}} t_i F_k. \qquad (*)$$

Since S spans $P^{\mathbf{G}}_{m+1}$, there exists a linearly independent subset B of S maximal with respect to $[B] \cap J^{\mathbf{G}}_{m+1} = \{0\}$, where $[B]$ is the linear span of B. Thus there is a $\mathbf{G}$-submodule C_{m+1} such that $P_{m+1} = J_{m+1} \oplus C_{m+1}$ and $C^{\mathbf{G}}_{m+1}$ has basis B. ■

COROLLARY *If $m > q_{\mathbf{G}}$ the method in Theorem* 6 *will work if the set S is replaced by the set $T = \{t_i t_j\colon i, j = 1, \ldots, k, \deg(t_i t_j) = m + 1\}$.*

Proof We prove this by induction on $m - q_{\mathbf{G}}$. We assume the corollary true up through degree m. By the theorem, we will be done if we can show that $P^{\mathbf{G}}_{m+1}$ is spanned by $J^{\mathbf{G}}_{m+1} \cup T$. From formula $(*)$ in the proof of the theorem, we know that

$$J^{\mathbf{G}}_{m+1} = \bigoplus_{\substack{k \geqslant 1 \\ d_i + k = m+1}} t_i F_k.$$

Furthermore, we know from Theorem 1 and $m > q_{\mathbf{G}}$ that $P^{\mathbf{G}}_0, \ldots, P^{\mathbf{G}}_m$ generate $P^{\mathbf{G}}$ and that $P^{\mathbf{G}}_0 \oplus \cdots \oplus P^{\mathbf{G}}_m = \bigoplus_{d_i + k \leqslant m} t_i F_k$. Thus $P^{\mathbf{G}}_{m+1}$ is spanned by $J^{\mathbf{G}}_{m+1} \cup D$ where $D = \{t_{i_1} \cdots t_{i_s}\colon 1 \leqslant i_j \leqslant k, \sum_j d_{i_j} = m + 1\}$. Now consider $t_{i_1} \cdots t_{i_s}$, one of the elements of D. Then $\deg(t_{i_2} \cdots t_{i_s}) < m + 1$. Since we have a partial completion up to degree m, we know that $t_{i_2} \cdots t_{i_s} = \sum c_{jr} t_j p_r$, where p_r is a monomial in the f_i's and $c_{jr} \in \mathbb{R}$. Thus, since $t_j p_r$ is in J when $\deg p_r > 0$, we can replace the spanning set D by the set $T = \{t_i t_j\colon\ 1 \leqslant i,\ j \leqslant k,\ \deg(t_i t_j) = m + 1\}$. The corollary follows. ■

Because of Theorem 6 and its corollary, a set of transients in a partial completion of $f_1, \ldots, f_n$ up to degree $m > q_{\mathbf{G}}$ is called *essential*. The remaining transients in a full completion of $f_1, \ldots, f_n$ can thus be chosen to be products of the essential transients.

7. GOOD POLYNOMIAL BASIS: SUBGROUPS AND REDUCIBLE REPRESENTATIONS

Let **G** have a good polynomial basis $f_1, \ldots, f_n$ (free), $t_1, \ldots, t_k$ (transients). Here is how to use this to find a good polynomibal basis for a subgroup **H** of **G**.

THEOREM 7 *Let C be a complement in P relative to $f_1, \ldots, f_n$. A good polynomial basis for $P^{\mathbf{H}}$ then consists of $f_1, \ldots, f_n$ as the free invariants together with a homogeneous vector space basis of $C^{\mathbf{H}}$ as the transient invariants.*

Proof This follows directly from Theorem 5 because $P^{\mathbf{G}} \subseteq P^{\mathbf{H}}$ and because the **G**-module C is automatically an **H**-module. ■

A well-known example is the following. Let **G** denote the group of $n \times n$ permutation matrices and $\sigma_1, \ldots, \sigma_n$ the elementary symmetric functions in the variables $X_1, \ldots, X_n$. Then for $P = \mathbb{R}[X_1, \ldots, X_n]$, $P^{\mathbf{G}} = \mathbb{R}[\sigma_1, \ldots, \sigma_n]$. Thus $\sigma_1, \ldots, \sigma_n$ is a good polynomial basis. There are no transients in this basis; they are all free.

Furthermore, if **H** is the subgroup of **G** corresponding to the even permutations (**H** is isomorphic to the alternating group) and if $\delta = \prod_{1 \leqslant i < j \leqslant n} (X_i - X_j)$, then $P^{\mathbf{H}} = \mathbb{R}[\sigma_1, \ldots, \sigma_n, \delta]$ and $\sigma_1, \ldots, \sigma_n, \delta$ is a good polynomial basis for **H** with the σ_i's free and δ the single transient.

Now assume ρ and σ are real representations of an abstract group **G** with respective good polynomial bases $f_1, \ldots, f_n$ (free), $t_1, \ldots, t_k$ (transient) and $g_1, \ldots, g_m$ (free), $u_1, \ldots, u_l$ (transient).

THEOREM 8 *Let C be a complement in $P(\rho)$ relative to $f_1, \ldots, f_n$ and D a complement in $P(\sigma)$ relative to $g_1, \ldots, g_m$. Let C' (respectively D') be a* **G**-*submodule of C (respectively D) such that $C = C^{\mathbf{G}} \oplus C'$ (respectively $D = D^{\mathbf{G}} \oplus D'$). Then a good polynomial basis for $P(\rho \oplus \sigma)$ consists of $f_1, \ldots, f_n, g_1, \ldots, g_m$ as free invariants and as transients all of the following*

(a) $t_1, \ldots, t_k, u_1, \ldots, u_l$;
(b) $t_i u_j$, $1 \leqslant i \leqslant k$, $1 \leqslant k \leqslant l$;

(c) *a homogeneous vector space basis of* $(C'D')^{\mathbf{G}}$*-called* cross-term *transients.*

Proof Let $F(\rho) = \mathbb{R}[f_1, \ldots, f_n]$ and $F(\sigma) = \mathbb{R}[g_1, \ldots, g_m]$. Then $P(\rho) \cong F(\rho) \otimes C$ and $P(\sigma) \cong F(\sigma) \otimes D$. From this and Theorem 2,

$$P(\rho \otimes \sigma) = P(\rho)P(\sigma) \simeq P(\rho) \otimes P(\sigma) \simeq (F(\rho) \otimes F(\sigma)) \otimes (C \otimes D).$$

But also $\mathbb{R}[f_1, \ldots, f_n, \ g_1, \ldots, g_m] = F(\rho)F(\sigma) \simeq F(\rho) \otimes F(\sigma)$ and $CD \simeq C \otimes D$. The theorem follows. ∎

Remark Finding a basis for $(C'D')^{\mathbf{G}}$ can be accomplished using the method of Proposition 4 part (1) with C' replacing $H(\mu)$ and D' replacing $H(\nu)$.

Finally, we turn to the description of a general method for finding a good polynomial basis for $P(\bigoplus^m \rho)^{\mathbf{G}}$ using earlier parts of this paper.

We recall the notation of Section 5 where the variables for $P(\bigoplus^m \rho)$ are denoted $X_i^{(j)}$, $i = 1, \ldots, n$; $j = 1, \ldots, m$ and the action of $\mathbf{g} \in \mathbf{G}$ on $X_i^{(j)}$ is given by $\mathbf{g}X_i^{(j)} = \sum_k a_{ki} X_k^{(j)}$. We extend this notation as follows. Suppose $P(\rho)$ is the polynomial ring in the variables $X_1, \ldots, X_n$ and that the free invariants in a good polynomial basis for $P(\rho)^{\mathbf{G}}$ are $f_1, \ldots, f_n$. Then let $f_i^{(j)} = f_i(X_1^{(j)}, \ldots, X_n^{(j)})$ $(i = 1, \ldots, n; j = 1, \ldots, m)$.

THEOREM 9 *Let* $m = n - 1$ *if* $\mathbf{G}$ *is a rotation group. Let* $m = n$ *otherwise. Let* $k \geqslant m$. *Then a good polynomial basis for* $P(\bigoplus^k \rho)^{\mathbf{G}}$ *can be chosen as follows.*

(1) *Choose the free invariants for* $P(\bigoplus^k \rho)^{\mathbf{G}}$ *to be the* $f_i^{(j)}$*'s.*
(2) *Assume the free invariants of a good polynomial basis for* $P(\bigoplus^m \rho)^{\mathbf{G}}$ *are the* $f_i^{(j)}$*'s* $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$. *Use Theorem* 8 *to complete them up to degree* $s = q_{\mathbf{G}}(\bigoplus^m \rho) + 1$.
(3) *Let* U *be the set of products* tf *such that* (i) *either* t *is a transient of degree* $\leqslant s$ *obtained in* (2) *(including* $t = t_0 = 1$) *or, in case* $\mathbf{G}$ *is a rotation group and* $n \leqslant s$, $t = \det(X_i^{(j)})_{1 \leqslant i,j \leqslant n}$; (ii) f *is a monomial in the* $f_i^{(j)}$*'s,* $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ *with* $\deg f \leqslant s - \deg t$.

Then $\mathrm{Pol}(U)$ *spans* $(\sum_i^s P_i(\bigoplus^k \rho))^{\mathbf{G}}$. *Use* $S = \mathrm{Pol}(U)$ *in Theorem* 6 *to complete* $\{f_i^{(j)}: 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k\}$ *up to degree* s.

8. GOOD POLYNOMIAL BASIS FOR RELATIVE INVARIANTS

In this section we will show that the span of the set of real relative invariants in P is equal to the ring of invariants in P associated with a certain subgroup of $\mathbf{G}$. Thus a good polynomial basis for relative invariants makes sense and finding one will be no more difficult than finding a good polynomial basis for a subgroup (Theorem 7).

If p is a relative invariant, then there exists a one-dimensional real representation $\lambda\colon \mathbf{G} \to \{\pm 1\}$ so that, for all $\mathbf{g} \in \mathbf{G}$, $\mathbf{g}p = \lambda(\mathbf{g})p$. In this case we call p a *relative invariant with weight* λ. The set $P^\lambda \subseteq P$ denotes the subspace of all relative invariants with weight λ. If $P_m^\lambda = P_m \cap P^\lambda$, then certainly $P^\lambda = \sum P_m^\lambda$.

The set of all linear characters of $\mathbf{G}$ forms a group, under pointwise multiplication, denoted by Λ. By definition, the set of all relative invariants is the union of the P^λ as λ ranges over Λ. We denote the span of this set by P^Λ so that $P^\Lambda = \sum_{\lambda \in \Lambda} P^\lambda$. The set P^Λ is an algebra that is nicely characterized by the following theorem.

THEOREM 10 *Let* $\hat{\mathbf{G}} = \{\mathbf{g} \in \mathbf{G}\colon \lambda(\mathbf{g}) = +1$ *for all* $\lambda \in \Lambda\}$. *Then* $\hat{\mathbf{G}}$ *is a normal subgroup of* $\mathbf{G}$ *and* $P^\Lambda = P^{\hat{\mathbf{G}}}$.

Proof We will prove this theorem in three stages:

(1) The set $\hat{\mathbf{G}}$ is a normal subgroup of $\mathbf{G}$.

For every $\lambda \in \Lambda$, $\ker \lambda$ is a normal subgroup. Furthermore, the intersection of normal subgroups is normal. This proves (1).

(2) $\mathbf{G}/\hat{\mathbf{G}} \simeq \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$. ($\mathbf{Z}_2$ denotes the integers modulo 2.)

Let $\mathbf{G}^2$ be the subgroup of $\mathbf{G}$ generated by the squares of elements in $\mathbf{G}$, i.e. $x \in \mathbf{G}^2$ iff there exist $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n \in \mathbf{G}$ with

$$x = \mathbf{g}_1^2\mathbf{g}_2^2 \cdots \mathbf{g}_n^2.$$

The subgroup $\mathbf{G}^2$ is normal because, for $\mathbf{h} \in \mathbf{G}$,

$$\mathbf{h}\mathbf{g}_1^2 \cdots \mathbf{g}_n^2\mathbf{h}^{-1} = (\mathbf{h}\mathbf{g}_1\mathbf{h}^{-1})^2 \cdots (\mathbf{h}\mathbf{g}_n\mathbf{h}^{-1})^2.$$

Furthermore, $\mathbf{G}^2 \subseteq \hat{\mathbf{G}}$ because, for $\lambda \in \Lambda$,

$$\lambda(\mathbf{g}_1^2 \cdots \mathbf{g}_n^2) = \lambda(\mathbf{g}_1)^2 \cdots \lambda(\mathbf{g}_n)^2 = 1.$$

Next, $\mathbf{a} \in \mathbf{G}/\mathbf{G}^2$ implies $\mathbf{a}^2 = 1 \in \mathbf{G}/\mathbf{G}^2$. For, if $\mathbf{a} = \mathbf{g}\mathbf{G}^2$, then $\mathbf{a}^2 = \mathbf{g}\mathbf{G}^2\mathbf{g}\mathbf{G}^2 = \mathbf{g}\mathbf{g}(\mathbf{g}^{-1}\mathbf{G}^2\mathbf{g})\mathbf{G}^2 = \mathbf{g}^2\mathbf{G}^2\mathbf{G}^2 = \mathbf{G}^2$. Thus $\mathbf{G}/\mathbf{G}^2 \simeq \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$.

Finally, to show $\hat{\mathbf{G}} \subseteq \mathbf{G}^2$, we note that if $\mathbf{G}^2 = \mathbf{G}$ then we are done. In case $\mathbf{G}^2 \neq \mathbf{G}$, we let $\mathbf{g} \in \mathbf{G}$ with $\mathbf{g} \notin \mathbf{G}^2$. Thus, if $\phi: \mathbf{G} \to \mathbf{G}/\mathbf{G}^2$ is the canonical epimorphism, then $\phi(\mathbf{g}) \neq 1 \in \mathbf{G}/\mathbf{G}^2$. Hence, there exists $\mu \in \Lambda(\mathbf{G}/\mathbf{G}^2)$ with $(\mu \circ \phi)(\mathbf{g}) = -1$. But $\mu \circ \phi \in \Lambda = \Lambda(\mathbf{G})$ and therefore $\mathbf{g} \notin \hat{\mathbf{G}}$. Thus $\hat{\mathbf{G}} \subseteq \mathbf{G}^2$ so that from what we had before, $\hat{\mathbf{G}} = \mathbf{G}^2$ and therefore $\mathbf{G}/\hat{\mathbf{G}} \simeq \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$. This proves (2).

(3) $P^\Lambda = P^{\hat{\mathbf{G}}}$.

It is clear that $P^\Lambda \subseteq P^{\hat{\mathbf{G}}}$. To show the inclusion in the other direction, we first show that $P_m^{\hat{\mathbf{G}}}$ is an invariant subspace of P_m under the action of $\mathbf{G}$. For, if $p \in P_m^{\hat{\mathbf{G}}}$, $\mathbf{g} \in \mathbf{G}$ and $\mathbf{h} \in \hat{\mathbf{G}}$, we have $\mathbf{g}^{-1}\mathbf{h}\mathbf{g} = \mathbf{h}' \in \hat{\mathbf{G}}$ ($\hat{\mathbf{G}}$ is a normal subgroup) so that $\mathbf{h}(\mathbf{g}p) = \mathbf{g}\mathbf{h}'p = \mathbf{g}p$. Thus $\mathbf{g}p \in P_m^{\hat{\mathbf{G}}}$.

As a consequence of the fact that $P_m^{\hat{\mathbf{G}}}$ is a $\mathbf{G}$-submodule of P_m, we note that the action of $\mathbf{G}$ on $P_m^{\hat{\mathbf{G}}}$ corresponds to a representation of $\mathbf{G}$ each of whose irreducible constituents η has the property that $\hat{\mathbf{G}} \subseteq \ker \eta$. One may then regard η as an irreducible representation of $\mathbf{G}/\hat{\mathbf{G}} \simeq \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$ so that therefore $\eta(\mathbf{g}) = \pm 1$ for every $\mathbf{g} \in \mathbf{G}$. Thus $P_m^{\hat{\mathbf{G}}} \subseteq P^\Lambda$ and the proof of the theorem is complete. ■

The notion of good polynomial basis makes sense for $P^{\hat{\mathbf{G}}}$. However, since not all elements of $P^{\hat{\mathbf{G}}}$ are relative invariants (the sum of two relative invariants is not necessarily a relative invariant), in order that a good polynomial basis for $P^{\hat{\mathbf{G}}}$ be useful, each of its elements must be in P_m^λ for some m and λ. A good polynomial basis for $P^{\hat{\mathbf{G}}}$ having this latter property will be called a Good Polynomial Basis for Relative Invariants (GPBRI for short).

One way to find a GPBRI once a good polynomial basis for $P^{\mathbf{G}}$ is known is given by Theorem 6 (since $\hat{\mathbf{G}}$ is a subgroup of $\mathbf{G}$) with the following modification: choose the homogeneous vector-space basis for $C^{\hat{\mathbf{G}}}$ to be relative invariants for $\mathbf{G}$.

Since $\hat{\mathbf{G}}$ is a group and $P^\Lambda = P^{\hat{\mathbf{G}}}$, facts in Theorem 1 about the invariant ideal and its complement (relative to $\hat{\mathbf{G}}$) can also be used to find a GPBRI. In particular, if $I(\mathbf{G})$ and $I(\hat{\mathbf{G}})$ denote the invariant ideals and $Q(\mathbf{G})$ and $Q(\hat{\mathbf{G}})$ denote complements relative to $\mathbf{G}$ and $\hat{\mathbf{G}}$ respectively, then we have $P^{\hat{\mathbf{G}}} \supseteq P^{\mathbf{G}}$, $Q(\mathbf{G}) \supseteq Q(\hat{\mathbf{G}})$, and $I(\hat{\mathbf{G}}) \supseteq I(\mathbf{G})$.

Recall that $q_{\mathbf{G}}$ (respectively $q_{\hat{\mathbf{G}}}$) is the smallest m such that $Q_m(\mathbf{G}) = 0$ (respectively $Q_m(\hat{\mathbf{G}}) = 0$). Thus $q_{\mathbf{G}} \geqslant q_{\hat{\mathbf{G}}}$ and the essential transients in a GPBRI can be chosen to be of degree $\leqslant q_{\hat{\mathbf{G}}} + 1$. This improvement on the bound $q_{\mathbf{G}} + 1$ on the degrees of a generating set for invariants may make it easier to seek relative invariants than to seek absolute invariants.

Furthermore, $\hat{\mathbf{G}}$ is always a rotation group so that the conclusions of Proposition 5 would hold, making computations even simpler (cf. Theorem 9).

Now let $\mathbf{G}$ be an abstract group, $\Lambda_{\mathbf{G}}$ be the group of all linear characters of $\mathbf{G}$ and $\hat{\mathbf{G}} = \{\mathbf{g} \in \mathbf{G}: \lambda(\mathbf{g}) = +1 \text{ for all } \lambda \in \Lambda_{\mathbf{G}}\}$. If ρ, σ are two representations of $\mathbf{G}$, the following theorem tells us how to find the relative invariants of $\rho \oplus \sigma$.

THEOREM 11 *If* $\phi: \mathbf{G} \to \mathbf{H}$ *is an epimorphism of groups, then* $\phi(\hat{\mathbf{G}}) = \hat{\mathbf{H}}$. *In particular,* $\sigma(\hat{\mathbf{G}}) = \widehat{\sigma(\mathbf{G})}$, $\rho(\hat{\mathbf{G}}) = \widehat{\rho(\mathbf{G})}$ *and* $(\sigma \oplus \rho)(\hat{\mathbf{G}}) = \widehat{(\sigma \oplus \rho)(\mathbf{G})}$. *Thus a good polynomial basis for relative invariants for* $\sigma \oplus \rho$ *can be found using Theorem* 8 *once they are known for* σ *and* ρ.

Proof From the proof of (2) within the proof of Theorem 10, $\hat{\mathbf{G}} = \mathbf{G}^2$ and $\hat{\mathbf{H}} = \mathbf{H}^2$. That $\sigma(\mathbf{G}^2) = \mathbf{H}^2$ follows easily from the fact that σ is an epimorphism. The theorem follows. ∎

Remark According to Theorem 11, to find a good polynomial basis for $\mathbf{G}$-relative invariants in $P(\rho \oplus \sigma)$, one may use Theorem 8 to compute a good polynomial basis for $P(\rho \oplus \sigma)^{\hat{\mathbf{G}}}$. However, a certain amount of care must be exercised in using Theorem 8 in order to guarantee that the cross-term transients (for $\hat{\mathbf{G}}$) are also relative invariants for $\mathbf{G}$. Here is a way to proceed. Assume the notation of Theorem 8, that $f_1, \ldots, f_n, t_1, \ldots, t_k$ form a good polynomial basis for $P(\rho)^{\hat{\mathbf{G}}}$, and that these are also relative invariants for $\mathbf{G}$. (Here, as in the discussion below, we focus on $P(\rho)$ and $f_1, \ldots, f_n, t_1, \ldots, t_k$ but assume that similar properties hold for $P(\sigma)$ and $g_1, \ldots, g_m, u_1, \ldots, u_l$.) Let J be the ideal generated by the f_i's. We claim that J_m is a $\mathbf{G}$-module as well as a $\hat{\mathbf{G}}$-module and that C can be chosen so that $C_m = P_m \cap C$ is a $\mathbf{G}$-module as well. We show this by induction on m. It is certainly true for $m = 1$. Assuming it is true up to $m = k$, we note that by formula $(*)$ in the proof of Theorem 6 $J_{m+1} = \bigoplus_{h+k=m+1} C_h F_k$. By the inductive hypothesis F_k and C_h are $\mathbf{G}$-modules. Thus so is J_{m+1}. Therefore, there is also a $\mathbf{G}$-module C_{m+1} such that $P_{m+1} = J_{m+1} \oplus C_{m+1}$.

Now, this construction can be made consistent with the fact that $C^{\hat{\mathbf{G}}}$ has basis $t_1, \ldots, t_k$—all relative invariants for $\mathbf{G}$. Hence $C^{\hat{\mathbf{G}}}$ is a $\mathbf{G}$-submodule of C and therefore $C^{\hat{\mathbf{G}}}$ has a $\mathbf{G}$-module complement C' in C. We assume that this is the C' of Theorem 8. We also assume that the D' (for $P(\sigma)$) of Theorem 8 is a $\mathbf{G}$-module. Consequently, $C'D'$ is also a $\mathbf{G}$-

module so that $(C'D')^{\hat{\mathbf{G}}}$ has a basis consisting of relative invariants for **G**. These are the desired cross-term transients.

9. GOOD POLYNOMIAL BASES: TWISTED REPRESENTATIONS

Let **G** be an abstract group, λ a real linear character of **G** and $\rho\colon \mathbf{G} \to \mathbf{O}(n)$ a representation. By $\lambda\rho$ denote the representation defined by $(\lambda\rho)(\mathbf{g}) = \lambda(\mathbf{g})\rho(\mathbf{g})$ and call it *the representation ρ twisted by λ* (or just a *twisted* representation, if the context is clear).

The following theorem shows how to find a GPBRI for a representation provided a GPBRI is known for a representation closely related to it by twisting.

THEOREM 12 *Let $\rho_1, \ldots, \rho_k$ be orthogonal representations of* **G** *and $\rho_1 \oplus \cdots \oplus \rho_k$ their direct sum. Let $\mu_1, \ldots, \mu_k$ be real linear characters of* **G**. *If the variables of ρ_i and $\mu_i\rho_i$ (all i) are identified in the obvious way, then a GPBRI for $\rho_1 \oplus \cdots \oplus \rho_k$ is also a GPBRI for $\mu_1\rho_1 \oplus \cdots \oplus \mu_k\rho_k$.*

Let $X_1^{(i)}, \ldots, X_{n_i}^{(i)}$ be the variables of $P(\rho_i)$ and $Y_1^{(i)}, \ldots, Y_{n_i}^{(i)}$ the variables of $P(\mu_i\rho_i)$ so that $\mathbf{g}X_j^{(i)} = \sum_k a_{kj}^{(i)}X_k^{(i)}$ and $\mathbf{g}Y_j^{(i)} = \sum_k \mu_i(\mathbf{g})a_{kj}^{(i)}Y_k^{(i)}$. The map $\psi_i\colon P(\rho_i) \to P(\mu_i\rho_i)$ defined by $\psi_i(X_j^{(i)}) = Y_j^{(i)}$ $(j = 1, \ldots, n_i)$ is what is meant by "identifying variables in the obvious way". Before proving the theorem we state and prove

LEMMA *If $p = p(X_j^{(i)}) \in P_{m_1}(\rho_1) \cdots P_{m_k}(\rho_k)$ and $q = p(Y_j^{(i)})$, then $q \in P_{m_1}(\mu_1\rho_1) \cdots P_{m_k}(\mu_k\rho_k)$ and $\mathbf{g}q = \mu_1(\mathbf{g})^{m_1} \cdots \mu_k(\mathbf{g})^{m_k}(\mathbf{g}p)|_{X=Y}$.*

Proof of Lemma We first show the lemma in case $k = 1$. So we suppress the superscripts on the variables. If $p \in P_m(\rho)$ and p is a monomial in the X_i's, then it is easy to see that, if $q = p(Y_1, \ldots, Y_n)$, then $\mathbf{g}q = \mu(\mathbf{g})^m(\mathbf{g}p)|_{X=Y}$. Extending by linearity, we see that for any $p \in P_m(\rho)$

$$\mathbf{g}q = \mu(\mathbf{g})^m(\mathbf{g}p)|_{X=Y}.$$

Now suppose that $p_i \in P_{m_i}(\rho_i)$ for $i = 1, \ldots, k$. Let $q_i = p_i(Y_1^{(i)}, \ldots, Y_{n_i}^{(i)})$. Then, by the preceding paragraph,

$$\mathbf{g}(q_1 \cdots q_k) = \prod_i \mathbf{g}q_i$$

$$= \prod_i \left[\mu_i(\mathbf{g})^{m_i}(\mathbf{g}p_i)|_{X^{(i)}=Y^{(i)}}\right].$$

But the latter is equal to

$$(\prod \mu_i(\mathbf{g})^{m_i})(\mathbf{g}(p_1 \cdots p_k))|_{X=Y}.$$

Extending this by linearity to all p in $P_{m_1}(\rho_1) \cdots P_{m_k}(\rho_k)$, we have the lemma. ■

Proof of Theorem Let p be a relative invariant in $P_m(\rho_1 \oplus \cdots \oplus \rho_k)$. Then $p = \sum p_{m_1 \cdots m_k}$ where $p_{m_1 \cdots m_k} \in P_{m_1}(\rho_1) \cdots P_{m_k}(\rho_k)$ by Theorem 2. Since the latter spaces are $\mathbf{G}$-modules and their direct sum $P_m(\rho_1 \oplus \cdots \oplus \rho_k)$, it must be the case that each $p_{m_1 \cdots m_k}$ is also a relative invariant.

Thus, without loss of generality, we may assume that $p \in P_{m_1}(\rho_1) \cdots P_{m_k}(\rho_k)$ for some $m_1, \ldots, m_k$. Since $\mathbf{g}p = \lambda(\mathbf{g})p$ for some $\lambda \in \Lambda$. we have by the lemma that, when $q \in P_{m_1}(\mu_1 \rho_1) \cdots P_{m_k}(\mu_k \rho_k)$ corresponds to p,

$$\begin{aligned} \mathbf{g}q &= \mu_1(\mathbf{g})^{m_1} \cdots \mu_k(\mathbf{g})^{m_k}(\mathbf{g}p)|_{X=Y} \\ &= \mu_1(\mathbf{g})^{m_1} \cdots \mu_k(\mathbf{g})^{m_k}(\lambda(\mathbf{g})p)|_{X=Y} \\ &= \mu_1(\mathbf{g})^{m_1} \cdots \mu_k(\mathbf{g})^{m_k}\lambda(\mathbf{g})q. \end{aligned}$$

Thus q is also a relative invariant. ■

Another useful theorem in the same spirit as Theorem 12 is the following.

THEOREM 13 *For every i, let $X^{(i)}$ be the polynomial variable corresponding to the linear representation λ_i. Then*

(1) *$X^{(1)}, \ldots, X^{(m)}$ are the free elements of a GPBRI for $P(\lambda_1 \oplus \cdots \oplus \lambda_m)$ (there are no transients in this basis);*
(2) *if $f_1, \ldots, f_k$ are the free elements of a GPBRI for $P(\sigma)$, then $\{X^{(1)}, \ldots, X^{(m)}\} \cup \{f_1, \ldots, f_k\}$ are free elements of a GPBRI for $P((\bigoplus_{i=1}^m \lambda_i) \oplus \sigma)$; the transients in this GPBRI are the transients from the GPBRI for $P(\sigma)$.*

Finally, we combine Theorems 4, 9 and 12 in the following.

THEOREM 14 *Let $\rho: \mathbf{G} \to \mathbf{O}(n)$ be a representation and $\mu_1, \ldots, \mu_k$ real linear characters of $\mathbf{G}$. Then a GPBRI for $P(\bigoplus_{i=1}^k \mu_i \rho)$ can be chosen as follows.*

(a) *Let $f_1, \ldots, f_n$ be the free invariants for a good polynomial basis of $P(\rho)^{\mathbf{G}}$. Let C be a complement in $P(\rho)$ relative to $f_1, \ldots, f_n$. Let*

$\lambda_1, \ldots, \lambda_l$ be all the real linear characters of **G** *and $C_{\lambda_i} = C \cap P(\rho)_{\lambda_i}$, $1 \leqslant i \leqslant l$. Then there is $C' \subseteq C$ so that $C = C_{\lambda_1} \oplus \cdots \oplus C_{\lambda_l} \oplus C'$ (**G**-module direct sum). The elements in the union, $\bigcup_i$ (basis for C_{λ_i}), form the transients of a good polynomial basis for $P(\rho)^{\hat{\mathbf{G}}}$. The free invariants of this basis are $f_1, \ldots, f_n$. This good polynomial basis for $P(\rho)^{\hat{\mathbf{G}}}$ is also a GPBRI for $P(\rho)$.*

(b) *Use Theorem* 9 *with $m = n - 1$ together with the Remark following Theorem* 11 *to construct a GPBRI for $P(\bigoplus^{n-1} \rho)$ in the form of a completion of the $f_i^{(j)}$'s $(1 \leqslant i \leqslant n, 1 \leqslant j < n - 1)$ up to degree $s = q_{\mathbf{G}}(\bigoplus^{n-1} \rho) + 1$.*

(c) *For the basis $X_1, \ldots, X_n$ of ρ, suppose $\mathbf{g}X_j = \sum_h a_{hj}(\mathbf{g})X_h$ for all $\mathbf{g} \in \mathbf{G}$. Then, for each $i = 1, \ldots, k$, choose a basis $Y_1^{(i)}, \ldots, Y_n^{(i)}$ for the twisted representation $\mu_i \rho$ so that $\mathbf{g}Y_j^{(i)} = \sum_h \mu_i(\mathbf{g})a_{hj}(\mathbf{g})Y_h^{(i)}$. Let $\varphi: P(\bigoplus_{i=1}^k \rho) \to P(\bigoplus_{i=1}^k \mu_i \rho)$ be the unitary $\mathbb{R}$-algebra homomorphism defined by $\varphi(X_j^{(i)}) = Y_j^{(i)}$. Then φ(GPBRI in* (b)) $=$ *GPBRI for $P(\bigoplus_{i=1}^k \mu_i \rho)$.*

In the case when **G** is a crystallographic point group, the representations of **G** can be decomposed into three or fewer representations of the form $\bigoplus^k \mu_i \rho$ where ρ is an irreducible representation. Furthermore, there are six necessary ρ's, each of degree 3 or less. Thus Theorem 14 is very useful in the study of such groups. This is exploited in [1].

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