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Relative invariants of crystallographic point groups

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Abstract. The authors describe an algorithm for constructing a good polynomial basis for the relative invariants associated with any real representation of a crystallographic point group (CPG). The novelty here is the universality of the algorithm. It depends on the fact that a typical real representation of a CPG has a particularly simple form and has a close relationship to a small number of low-dimensional matrix groups whose invariants are known and well behaved.

1. Introduction

Let $\mathbf{G}$ be a group of real $n\times n$ matrices. If $p(X_1, \ldots, X_n)$ is a polynomial in the variables $X_1, \ldots, X_n$ with real coefficients and $g=(a_{ij})$ is an element of $\mathbf{G}$, then we let $\mathbf{G}$ act on $P=R[X_1, \ldots, X_n]$ by $gp(X_1, \ldots, X_n)=p(\Sigma a_{j1}X_j, \ldots, \Sigma a_{jn}X_j)$. The ring $P^{\mathbf{G}}$ of *invariants* of $\mathbf{G}$ consists of all polynomials p such that $gp=p$ for all g in $\mathbf{G}$.

We are interested in describing $P^{\mathbf{G}}$ in case $\mathbf{G}$ is the image group of some representation of a crystallographic point group (CPG). One may think of the elements of $P^{\mathbf{G}}$ as being thermodynamic potentials dependent on certain physical quantities and invariant under $\mathbf{G}$.

A convenient way of describing $P^{\mathbf{G}}$ is as

$$P^{\mathbf{G}}=\bigoplus_{k=0}^{d} t_k R[f_1, \ldots, f_n]$$

where $f_1, \ldots, f_n$ are algebraically independent homogenous polynomials, $t_0=1$ and $t_1, \ldots, t_d$ are other homogeneous invariants. The set of invariants $f_1, \ldots, f_n, t_1, \ldots, t_d$ form what is called a *good polynomial basis* (GPB) with $f_1, \ldots, f_n$ the *free* invariants and $t_1, \ldots, t_d$ the *transient* invariants. Good polynomial bases for selected CPGs are given in table 4.

What we will actually do in this paper is to describe the real relative invariants of P in case $\mathbf{G}$ is the image group of some real representation of a crystallographic point group. We will do this in a way very close to the description above, utilising the notion of *good polynomial basis for relative invariants* (GPBRI) as introduced in Gay and Ascher (1984). To explain this, recall that a polynomial p is a relative invariant for $\mathbf{G}$ if there exists a homomorphism $\lambda: \mathbf{G}\to\{+1, -1\}$ such that $gp=\lambda(g)p$ for all $g\in\mathbf{G}$. In this case

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p is a relative invariant with weight λ. All the relative invariants of weight λ are gathered into a single vector space P_λ. Let Λ denote the set of all such homomorphisms λ and set $P_\Lambda = \Sigma_{\lambda \in \Lambda} P_\lambda$. From Gay and Ascher (1984) the latter is a subalgebra equal to the algebra of polynomial invariants $P^{\hat{\mathbf{G}}}$ where $\hat{\mathbf{G}}$ is the subgroup of $\mathbf{G}$ equal to all g in $\mathbf{G}$ such that, for all λ in Λ, $\lambda(g) = +1$. A good polynomial basis for $P^{\hat{\mathbf{G}}}$, in the sense given above, is also a GPBRI for $\mathbf{G}$ if both the free and transient elements of the basis are also relative invariants for $\mathbf{G}$.

This paper follows the work of those who have described the standard, three-space variable invariants for the CPGs (Meyer 1954, Döring 1958, Döring 1960, Döring and Simon 1961, Smith *et al* 1963, Smith and Rivlin 1964, Killingbeck 1972, Patera *et al* 1978) as well as those who have displayed the invariants corresponding to the irreducible representations of the CPGs and to other special representations (Spencer 1971, McLellan 1974, Bickerstaff and Wybourne 1976). This paper is also a partial response to the call by Louis Michel in his Montreal lecture (1977) to describe the (relative) invariants of a representation given that the (relative) invariants of its irreducible constituents are known.

The mathematical tools on which this paper most heavily depend were developed in two papers: Kopský (1979a) and Gay and Ascher (1984). The latter was built, in turn, on ideas from Capelli (1887), Chevalley (1955), Sloane (1977), Solomon (1977) and Stanley (1979).

The remainder of the paper is organised as follows. In § 2 we will give a particularly simple description of the real representations of the CPGs and show how the notion of twisted representation is useful in this description. In § 3 we will establish notation and summarise the results that we will need from previous work; we will show how some of these ideas are connected with the CPGs. In § 4 we will apply the ideas of § 3 directly to three examples important for CPGs. The latter will play a central role in the description of the algorithm for constructing GPBRIs in § 5. Finally, in the appendix (§ 6), we will provide more information about the methods used to obtain the results of §§ 4 and 5.

2. Real representations of CPGS

In the previous section we discussed the invariants associated with a matrix group $\mathbf{G} \subseteq \mathrm{GL}(n)$. When $\mathbf{G}$ is a CPG and $\mathbf{G} \subseteq \mathbf{O}(3)$, then the natural invariants are considered by some as *the* invariants of $\mathbf{G}$. However, when thermodynamic potentials are considered, the situation is complicated and the language of invariants must be clarified.

2.1. First simplification: Abstract CGPS

Suppose we want to find a thermodynamic potential depending on given physical quantities $q_1, \ldots, q_n$ and invariant under a given CPG $\mathbf{G}$. The first step in the process is to determine which representation ρ of $\mathbf{G}$ corresponds to the quantities $q_1, \ldots, q_n$. (These quantities usually correspond more directly to a representation of $\mathbf{O}(3)$; the question of which representation ρ of $\mathbf{G}$ this latter corresponds to is considered in Bickerstaff and Wybourne (1976).) The thermodynamic potential is then an invariant (in the sense of § 1) of the matrix group $\rho(\mathbf{G})$. Thus we can replace the problem of finding the invariants of $\rho(\mathbf{G})$ by the problem of finding the invariants of $\rho(\mathbf{H})$ for every *abstract* CPG $\mathbf{H}$ and for every representation ρ of $\mathbf{H}$. Since the number of CPGS is 32 and the number of abstract CPGs is 17 (see also the remark at the end of this section), this is a considerable simplification.

By way of example, consider the four (concrete) CPGs 432, $\overline{43}m$, 4'32' and $\bar{4}'3m'$. Each is isomorphic to the group $\mathbf{s}_4$ of all permutations of four things. The abstract group $\mathbf{s}_4$ has five irreducible representations which we denote by A_1, A_2, E, T_1, and T_2 (following the notation of Bradley and Cracknell 1972 for the group 432). In table 1 we list the representations corresponding to electric polarisation P, magnetisation M and stress σ for each of the four CPGs. Thus to find a thermodynamic potential depending only on electric polarisation and invariant under 432, one looks at the invariants of the representation T_1 of $\mathbf{s}_4$. But to find a thermodynamic potential (depending only on electric polarisation) invariant under $\bar{4}'3m'$, one considers the invariants of the representation T_2 of $\mathbf{s}_4$. The simplification indicated above means that, when seeking a thermodynamic potential for one of these four CPGs, one ultimately looks at the invariants associated with some representation of $\mathbf{s}_4$.

Table 1.

	P	M	σ	P, M and σ
432	T_1	T_1	A_1+E+T_2	$A_1+E+2T_1+T_2$
$\overline{43}m$	T_2	T_1	A_1+E+T_2	$A_1+E+T_1+2T_2$
4'32'	T_1	T_2	A_1+E+T_2	$A_1+E+T_1+2T_2$
$\bar{4}'3m'$	T_2	T_2	A_1+E+T_2	A_1+E+3T_2

2.2. Second simplification: The 14 faithful irreps

The nature of the real representations of the abstract CPGs leads to a second simplification. Since every representation of a finite group is completely reducible, the main building blocks of our representations are the matrix groups $\rho(\mathbf{G})$ where $\mathbf{G}$ is an abstract CPG and ρ is an irreducible representation of $\mathbf{G}$. The simplifying fact here is that $\rho(\mathbf{G})$ is then one of fourteen matrix groups (arising from the fourteen faithful irreps of the abstract CPGs). We will frequently not distinguish these matrix groups from the representations whose image groups they are.

To take advantage of this favourable situation, we need a useful notation—one that is reasonably consistent with existing literature and one that distinguishes these 14 groups. Here are our notational assumptions.

(1) We denote the abstract cyclic, dihedral, symmetric and alternating groups by $\mathbf{c}_m$, $\mathbf{d}_m$, $\mathbf{s}_m$ and $\mathbf{a}_m$ respectively. All abstract CPGs can be created from these using the direct product of groups.

(2) Almost every abstract CPG has a unique, concrete CPG, consisting of pure rotations, to which it is isomorphic. For the representations of such groups, we use the notation of Bradley and Cracknell (1972), with some slight adjustments due to these facts:

(a) All our representations are real, whereas the tables mentioned above include non-real representations.

(b) The faithful irreps of $\mathbf{d}_3$ and $\mathbf{d}_4$ are denoted there by the same symbol.

Table 2 summarises the changes from the Bradley–Cracknell notation. Furthermore for the irreps of abstract direct product groups of the form $\mathbf{G}\times\mathbf{c}_2$ having no corresponding (concrete) CPG consisting of pure rotations, we adopt the convention

$$X \otimes \boldsymbol{B} = X_u$$

where X is an irrep of **G** and **B** is the faithful irrep of $\mathbf{c}_2$.

Finally, we list the 14 groups in table 3 using all these conventions.

Table 2. Notational conventions for the real irreducible representations of CPGs.

Abstract CPG	Bradley–Cracknell	This work
$\mathbf{c}_3$	${}^1E + {}^2E$	**e**
$\mathbf{d}_3$	A_1	**A**
	A_2	**B**
$\mathbf{c}_4$	${}^1E + {}^2E$	**f**
$\mathbf{c}_6$	${}^1E_1 + {}^2E_1$	$\mathbf{e}_1$
	${}^1E_2 + {}^2E_2$	$\mathbf{e}_2$
$\mathbf{d}_4$	A_1	**A**
	A_2	$\mathbf{B}_3$
	E	**F**
$\mathbf{a}_4$	A_1	**A**
	A_2	**B**
	${}^1E + {}^2E$	**e**
$\mathbf{s}_4$	A_1	**A**
	A_2	**B**
$\mathbf{d}_6$	A_1	**A**
	A_2	$\mathbf{B}_3$

Table 3. The 14 matrix groups arising from the 14 faithful irreps of the CPGs.

Matrix group (real faithful irrep of CPG)	Dimension of rep.	Isomorphic abstract group	Isomorphic crystallographic group of rotations	Geometric characterisation of matrix group
A	1	$\mathbf{c}_1$	1	
B	1	$\mathbf{c}_2$	2	
e	2	$\mathbf{c}_3$	3	rotational symmetries of triangle
E	2	$\mathbf{d}_3$	32	full symmetries of triangle
$\mathbf{e}_2$	2	$\mathbf{c}_6$	6	rotational symmetries of regular hexagon
$\mathbf{E}_1$	2	$\mathbf{d}_6$	622	full symmetries of regular hexagon
f	2	$\mathbf{c}_4$	4	rotational symmetries of squares
F	2	$\mathbf{d}_4$	422	full symmetries of square
T	3	$\mathbf{a}_4$	23	rotational symmetries of tetrahedron
$\mathbf{T}_1$	3	$\mathbf{s}_4$	432	rotational symmetries of cube
$\mathbf{T}_2$	3	$\mathbf{s}_4$	432	full symmetries of tetrahedron
$\mathbf{T}_u$	3	$\mathbf{a}_4 \times \mathbf{c}_2$	$(m3)$†	
$\mathbf{T}_{1u}$	3	$\mathbf{s}_4 \times \mathbf{c}_2$	$(m\bar{3}m)$†	full symmetries of cube
$\mathbf{T}_{2u}$	3	$\mathbf{s}_4 \times \mathbf{c}_2$	$(m\bar{3}m)$†	

† Not rotation groups.

2.3. *Third simplification: Reflection groups, rotation subgroups and twisted representations*

If $\mathbf{G} \subseteq \mathbf{O}(n)$ is a reflection group, then the set of all $g \in \mathbf{G}$ with $\det g = +1$ forms a subgroup of $\mathbf{G}$ called the *rotation subgroup* of $\mathbf{G}$. For example, in table 3, the matrix groups $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{T}_2$ are reflection groups and $\mathbf{e}, \mathbf{f}, \mathbf{T}$ are their respective rotation subgroups. Any irrep of a CPG must be one of these six or be one-dimensional or be one of the six "twisted" by a one-dimensional real representation (in a way we make precise below).

Let $\rho: \mathbf{G} \to \mathbf{O}(n)$ be a representation of $\mathbf{G}$, λ a one-dimensional (real) representation of $\mathbf{G}$. Denote by $\lambda\rho$ the representation $(\lambda\rho)(g) = \lambda(g)\rho(g)$ and call it the *representation ρ twisted by λ* (or just a *twisted* representation if the context is clear).

In table 3, the groups $\mathbf{T}_1, \mathbf{T}_{1u}, \mathbf{T}_{2u}$ are all twisted versions of $\mathbf{T}_2$; $\mathbf{T}_u$ is a twisted $\mathbf{T}$; $\mathbf{e}_2$ and $\mathbf{E}_1$ are twisted $\mathbf{e}$ and $\mathbf{E}$ respectively†.

2.4. *Fourth simplification: A general form for a real representation of an abstract* CPG

The fourth and final simplification of the original problem is related to the way the 14 faithful irreps can be combined to constitute a typical real rep of a CPG. This is described in the following.

Theorem 1. Let $\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n$ be one-dimensional real representations (not necessarily distinct) of an abstract CPG $\mathbf{G}$. Then a typical real representation of $\mathbf{G}$ must be one of the following types:

(1) $\lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_l$

(2) $\bigoplus_{i=1}^{l} \lambda_i + \bigoplus_{i=1}^{m} (\mu_j \varphi)$ where $\varphi = \mathbf{f}$ or $\varphi = \mathbf{F}$

(3) $\bigoplus_{i=1}^{l} \lambda_i + \bigoplus_{i=1}^{m} (\mu_j \eta) + \bigoplus_{k=1}^{n} (\nu_k \tau)$ where $\eta = \mathbf{e}$, $\tau = \mathbf{T}$ or

$\eta = \mathbf{E}$, $\tau = \mathbf{T}_2$.

It is understood that in (2) or (3) the set $\{\lambda_1, \ldots, \lambda_l\}$ might be vacuous, that in (3) either $\{\mu_1, \ldots, \mu_m\}$ or $\{\nu_1, \ldots, \nu_n\}$ might be vacuous (but not both).

Sketch of proof of theorem 1. Consider the following abstract groups

$$\mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{d}_4, \mathbf{d}_3 (= \mathbf{s}_3), \mathbf{a}_4, \mathbf{s}_4.$$

Every CPG is either isomorphic to one of these seven groups or to the direct product of one of these with one or more copies of $\mathbf{c}_2$. The real representations of any CPG can then be constructed from the representations of the seven groups above by knowing how representations for direct products are constructed from representations of the factors. Theorem 1 then follows from this and a knowledge of the irreducible representation of these seven groups.

† There is possibility for confusion here. To say '$\mathbf{V}$ is a twisted $\mathbf{W}$' means that there is a CPG $\mathbf{G}$ with two representations having image group $\mathbf{V}$ and $\mathbf{W}$ related as above. There may be a CPG $\mathbf{H}$ having $\mathbf{W}$ (as an image group) but not $\mathbf{V}$. The former group $\mathbf{G}$ has the appropriate, non-trivial twisting λ; the latter group $\mathbf{H}$ does not.

Remark. Theorem 1 is also true for an abstract crystallographic Schubnikov point group. The argument is the same. See Ascher and Janner (1965) for a list of the 122 concrete and 25 abstract crystallographic Schubnikov point groups. Thus the results of this paper are valid for this larger class of groups as well as for CPGS.

3. Background and notation

In this section we will set the stage for the rest of this paper by establishing additional notation and highlighting results from previous work that we will use. Facts that are recent are set apart and numbered from (3.1) to (3.9). These appeared in Gay and Ascher (1984); (3.4) also appeared in Kopský (1979a).

We assume that $\mathbf{G}$ is a finite subgroup of $\mathbf{O}(n)$ and that P is the associated polynomial ring. For a $\mathbf{G}$-module M denote by $M^{\mathbf{G}}$ the set of all $m \in M$ with $gm = m$ for all $g \in \mathbf{G}$. Let P_m, $P_m^{\mathbf{G}}$ denote the homogeneous elements of degree m in P, $P^{\mathbf{G}}$ respectively. Then $P = \Sigma\, P_m$ and $P^{\mathbf{G}} = \Sigma\, P_m^{\mathbf{G}}$.

3.1. Complements

Let I be the ideal in P generated by the invariant polynomials with no constant term. If $I_m = I \cap P_m$, then $I = \Sigma_m I_m$ and $I_m = \Sigma_{k=1}^{m} P_k^{\mathbf{G}} P_{m-k}$ (the latter is not necessarily a direct sum). For all k, the subspace $P_k^{\mathbf{G}} P_{m-k}$ is a $\mathbf{G}$-submodule. Thus I_m is also a $\mathbf{G}$-module and consequently it has a $\mathbf{G}$-submodule complement Q_m in P_m: $P_m = I_m \oplus Q_m$. Let $Q = \Sigma\, Q_m$ and call Q a *complement* to I in P. Then

(3.1) Q is finite dimensional and, if $Q_m = \{0\}$, $P^{\mathbf{G}}$ is generated by $P_1^{\mathbf{G}}, \ldots, P_m^{\mathbf{G}}$.

(By contrast, Noether's theorem (1916) states that if $|\mathbf{G}|$ is the order of $\mathbf{G}$, then $P^{\mathbf{G}}$ is generated by $P_1^{\mathbf{G}}, \ldots, P_{|\mathbf{G}|}^{\mathbf{G}}$.)

Complements for selected CPGS are shown in table 4. For each CPG, a good polynomial basis is also shown. For a given group $\mathbf{G}$ of matrices, a good polynomial basis plus a basis for the complement is what Kopský (1979a, b) calls a minimal extended integrity basis for $\mathbf{G}$. In that paper, adopting language of McLellan (1974), he uses the terms numerator and denominator invariants instead of free and transient invariants. The polynomials in table 4 also appear in Kopský (1979b).

3.2. **G***-Harmonic polynomials and reducible representations*

One choice for a complement can be described as follows.

Let $p \in P_m$ with $p = \Sigma\, \alpha_{i_1 \ldots i_m} X_{i_1} \ldots X_{i_m}$, $\alpha_{i_1 \ldots i_m} \in R$. Define $D_p = \Sigma\, \alpha_{i_1 \ldots i_m} (\partial/\partial X_{i_1}) \ldots (\partial/\partial X_{i_m})$. If p is not homogeneous, define D_p in the obvious manner. Then each D_p operates on P in the usual way. For all $m > 0$, let $H_m = \{q \in P_m : D_p(q) = 0$ for all $p \in P^{\mathbf{G}}\}$ and set $H = \Sigma\, H_m$. It is clear that $H = \{q \in P : D_p(q) = 0$ for all $p \in P^{\mathbf{G}}\}$. We call H the $\mathbf{G}$-*harmonic polynomials.* Then H is a complement to I (see Gay and Ascher 1984).

For a fixed orthogonal representation ρ of $\mathbf{G}$, denote by $P(\rho)$, $P(\rho)^{\mathbf{G}}$, $I(\rho)$, $H(\rho)$ the ring of polynomials, invariants, invariant ideal and harmonic polynomials respectively associated with the matrix group $\rho(\mathbf{G})$.

Table 4. Good polynomial basis and complement for selected irreducible representations of CPGs.

(*a*)

	$\rho(\mathbf{G})=\mathbf{E}$		$\rho(\mathbf{G})=\mathbf{e}$	
Polynomial degree k	Polynomial basis for Q_k	**G**-action on Q_k	Polynomial basis for Q_k	**G**-action on Q_k
0	1	**A**	same as	**A**
1	$\{u, v\}$	**E**	for	**e**
2	$\{u^2-v^2, -2uv\}$	**E**	$\rho(\mathbf{G})=\mathbf{E}$	**e**
3	$\{v^3-3vu^2\}$	**B**	$\varnothing$	0
4	$\varnothing$	0	$\varnothing$	0
Good polynomial basis	$p_1(u, v)=u^2+v^2$, $p_2(u, v)=u^3-3uv^2$(free)		p_1, p_2(free), $p_3(u, v)=v^3-3vu^2$(transient)	

(*b*)

	$\rho(\mathbf{G})=\mathbf{F}$		$\rho(\mathbf{G})=\mathbf{f}$	
Polynomial degree k	Polynomial basis for Q_k	**G**-action on Q_k	Polynomial basis for Q_k	**G**-action on Q_k
0	1	**A**	same as	**A**
1	$\{p, q\}$	**F**	for	**f**
2	$\{p^2-q^2\}, \{pq\}$	$\mathbf{B}_1\oplus\mathbf{B}_2$	$\rho(\mathbf{G})=\mathbf{F}$	$\mathbf{B}\oplus\mathbf{B}$
3	$\{p^3, q^3\}$	**F**		**f**
4	$\{pq(p^2-q^2)\}$	$\mathbf{B}_3$	$\varnothing$	0
5	$\varnothing$	0	$\varnothing$	0
Good polynomial basis	$r_1(p, q)=p^2+q^2$, $r_2(p, q)=p^4+q^4$(free)		r_1, r_2(free), $r_3(p, q)=pq(p^2-q^2)$(transient)	

(*c*)

	$\rho(\mathbf{G})=\mathbf{T}_2$		$\rho(\mathbf{G})=\mathbf{T}$	
Polynomial degree k	Polynomial basis for Q_k	**G**-action on Q_k	Polynomial basis for Q_k	**G**-action on Q_k
0	1	**A**		**A**
1	$\{x, y, z\}$	$\mathbf{T}_2$	same	**T**
2	$\{2z^2-x^2-y^2, \sqrt{3}(x^2-y^2)\}, \{yz, zx, xy\}$	$\mathbf{E}\oplus\mathbf{T}_2$	as	$\mathbf{e}\oplus\mathbf{T}$
3	$\{(y^2-z^2)x, \ldots\}$†, $\{(y^2+z^2)x, \ldots\}$†	$\mathbf{T}_1\oplus\mathbf{T}_2$	for	$\mathbf{T}\oplus\mathbf{T}$
4	$\{2x^2y^2-(x^2+y^2)z^2, -\sqrt{3}(x^2-y^2)z^2\}$, $\{(y^2-z^2)yz, \ldots\}$†	$\mathbf{E}\oplus\mathbf{T}_1$	$\rho(G)=T_2$	$\mathbf{e}\oplus\mathbf{T}$
5	$\{(y^2-z^2)x^3, \ldots\}$†	$\mathbf{T}_1$		**T**
6	$\{(x^2-y^2)(y^2-z^2)(z^2-x^2)\}$	**B**	$\varnothing$	0
7	$\varnothing$	0	$\varnothing$	0
Good polynomial basis	$q_1(x, y, z)=x^2+y^2+z^2$, $q_2(x, y, z)=xyz$, $q_3(x, y, z)=x^4+y^4+z^4$(free)		q_1, q_2, q_3(free), $q_4(x, y, z)=(x^2-y^2)(y^2-z^2)(z^2-x^2)$ (transient)	

† $\{p(x, y, z), \ldots\}$ means $\{p(x, y, z), p(z, x, y), p(y, z, x)\}$.

Let ρ and σ be two representations of **G** and $\rho \oplus \sigma$ their direct sum. Let $\bigoplus^n \rho$ denote

$$\underbrace{\rho \oplus \ldots \oplus \rho}_{n \text{ copies}}.$$

(3.2) $P_m(\rho \oplus \sigma)^{\mathbf{G}}$ is generated by $P_1(\rho \oplus \sigma)^{\mathbf{G}}, \ldots, P_{m-1}(\rho \oplus \sigma)^{\mathbf{G}}, P_m(\rho)^{\mathbf{G}}, P_m(\sigma)^{\mathbf{G}}$, and $\Sigma_{i=1}^{m-1}[H_i(\rho)H_{m-i}(\sigma)]^{\mathbf{G}}$.

3.3. *Complements relative to free invariants*

Let $f_1, \ldots, f_n$ be the free invariants of a good polynomial basis for $P^{\mathbf{G}}$. Then transient invariants for the basis can be found as follows. Let J be the ideal in P generated by the free invariants. Let $J_m = P_m \cap J$ so that, because the f_is are homogeneous, $J = \Sigma J_m$. Certainly, J_m is a **G**-submodule so that there exists a **G**-submodule C_m of P_m with $P_m = J_m \oplus C_m$. Let $C = \Sigma C_m$ and call C a *complement relative to* $f_1, \ldots, f_n$.

(3.3) The transients for the original polynomial basis form a homogeneous basis of $C^{\mathbf{G}}$ for some relative complement C. Conversely, given a relative complement C, a homogeneous basis for $C^{\mathbf{G}}$ together with $f_1, \ldots, f_n$ constitutes a good polynomial basis with the former transient and the latter free.

3.4. *Cross-term transients*

Now assume ρ and σ are real representations of an abstract group **G** with respective good polynomial bases $f_1, \ldots, f_n$ (free), $t_1, \ldots, t_k$ (transient) and $g_1, \ldots, g_m$ (free), $u_1, \ldots, u_l$ (transient).

Let C be a complement in $P(\rho)$ relative to $f_1, \ldots, f_n$ and D a complement in $P(\sigma)$ relative to $g_1, \ldots, g_m$. Let C' (respectively D') be a **G**-submodule of C (respectively D) such that $C = C^{\mathbf{G}} \oplus C'$ (respectively $D = D^{\mathbf{G}} \oplus D'$).

(3.4) A good polynomial basis for $P(\rho \oplus \sigma)$ consists of $f_1, \ldots, f_n, g_1, \ldots, g_m$ as free invariants and as transients all of the following:

(a) $t_1, \ldots, t_k, u_1, \ldots, u_l$,

(b) $t_i u_j$, $1 \leqslant i \leqslant k$, $1 \leqslant k \leqslant l$,

(c) a homogeneous vector space basis of $(C'D')^{\mathbf{G}}$ whose elements we will call *cross-term* transients.

3.5. *Relative invariants*

Recall from § 1 that if **G** is a finite group, then $\hat{\mathbf{G}} = \{g \in \mathbf{G}: \lambda(g) = +1 \text{ for all } \lambda \in \Lambda\}$ where Λ is the set of real, linear representations of **G**.

(3.5) If $\rho: \mathbf{G} \to \mathbf{O}(n)$ is a representation of **G**, then the algebra generated by the relative invariants of $\rho(\mathbf{G})$ is the same as the algebra of invariants of $\rho(\hat{\mathbf{G}})$. In other words, $\widehat{\rho(\mathbf{G})} = \rho(\hat{\mathbf{G}})$. Furthermore, $\rho(\mathbf{G}) \subseteq \mathbf{O}(n)$ implies that $\rho(\hat{\mathbf{G}})$ is a group of rotations.

The group $\hat{\mathbf{G}}$ corresponding to each of the 14 matrix groups **G** of table 3 are given in table 5.

Table 5.

Real irreps of CPGs	$\hat{\mathbf{G}}$
$\mathbf{A}, \mathbf{B}$	$\mathbf{A}$
$\mathbf{e}, \mathbf{E}, \mathbf{e}_2, \mathbf{E}_1$	$\mathbf{e}$
$\mathbf{f}, \mathbf{F}$	$\mathbf{B}\oplus\mathbf{B}$
$\mathbf{T}, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_u, \mathbf{T}_{1u}, \mathbf{T}_{2u}$	$\mathbf{T}$

3.6. Polarisation

Let $\rho:\mathbf{G}\to\mathbf{O}(n)$ be a representation of $\mathbf{G}$ with corresponding polynomial basis $X_1,\ldots,X_n$. For $\boldsymbol{h}\in\rho(\mathbf{G})$ suppose $\boldsymbol{h}X_i=\Sigma_j a_{ij}X_j$. For $l=1,\ldots,m$ let $X_1^{(l)},\ldots,X_n^{(l)}$ be polynomial variables with $\boldsymbol{h}X_i^{(l)}=\Sigma_j a_{ij}X_j^{(l)}$, i.e., $\rho(\mathbf{G})$ acts on $X_1^{(l)},\ldots,X_n^{(l)}$ just as it acts on $X_1,\ldots,X_n$. Furthermore, assume that the $X_i^{(l)}$, $i=1,\ldots,n$, $l=1,\ldots,m$ are algebraically independent forming a polynomial basis for the representation $\bigoplus^m\rho$. For $i,j=1,\ldots,m$ we define an operator D_{ij} on $P(\bigoplus^m\rho)$ by $D_{ij}p=\Sigma_k X_k^{(i)}(\partial/\partial X_k^{(j)})p$, called *polarisation of p with respect to* $(X_1^{(i)},\ldots,X_n^{(i)})$ *at* $(X_1^{(j)},\ldots,X_n^{(j)})$.

For a subset $S\subseteq P(\bigoplus^m\rho)$, we denote by $\mathrm{Pol}(S)$ the vector space span of those $q\in P(\bigoplus^m\rho)$ for which there exists $p\in S$ and integers $1\leqslant i_1,\ldots,i_k,j_1,\ldots j_k\leqslant m$ so that $q=D_{i_1j_1}\ldots D_{i_kj_k}p$. The main facts about these notions are as follows.

(3.6) Let $m\geqslant n=\dim\rho$. For a representation σ with harmonic polynomials $H(\sigma)$, let $q(\sigma)=\max_k\{H_k(\sigma)\neq 0\}$. Then we have

(a) $P(\bigoplus^m\rho)=\mathrm{Pol}(P(\bigoplus^n\rho))$.

(b) $P(\bigoplus^m\rho)^{\mathbf{G}}=\mathrm{Pol}(P(\bigoplus^n\rho)^{\mathbf{G}})$.

(c) $H(\bigoplus^m\rho)=\mathrm{Pol}(H(\bigoplus^n\rho))$.

(d) $q(\bigoplus^m\rho)=q(\bigoplus^n\rho)$.

(e) The operator $D_{i_1j_1}\ldots D_{i_kj_k}$ is a $\mathbf{G}$-module homomorphism.

(f) If $\rho(\mathbf{G})$ is a rotation group, then $H(\bigoplus^m\rho)=\mathrm{Pol}(H(\bigoplus^{n-1}\rho))$ and $q(\bigoplus^m\rho)=q(\bigoplus^{n-1}\rho)$. If $\alpha=\det(X_j^{(i)})_{1\leqslant i,j\leqslant n}$, then for $m\geqslant n-1$, $P(\bigoplus^m\rho)^{\mathbf{G}}$ is the linear span of the image of the set $P(\bigoplus^{n-1}\rho)^{\mathbf{G}}\cup\{\alpha\}$ under polar operators of the form $D_{i_1j_1}\ldots D_{i_kj_k}$ $(1\leqslant i_1,\ldots,i_k\leqslant m;\ 1\leqslant j_1,\ldots,j_k\leqslant n)$.

3.7. Completing a good polynomial basis

According to (3.3), if $f_1,\ldots,f_n$ are the free invariants and $t_1,\ldots,t_k$ some of the transient invariants of a good polynomial basis, then there are many possible choices for polynomials $s_1,\ldots,s_h$ so that $f_1,\ldots,f_n$ (free), $t_1,\ldots,t_k$, $s_1,\ldots,s_h$ (transient) is also a good polynomial basis. The latter is called a *completion* of the former. If $f_1,\ldots,f_n$ are the free invariants of a good polynomial basis and $t_1,\ldots,t_k$ are the transients of degree $\leqslant m$ for some completion of $f_1,\ldots,f_n$, then we say that $f_1,\ldots,f_n$, $t_1,\ldots,t_k$ is a *partial completion* of $f_1,\ldots,f_n$ *up to degree m*. Here is a method for obtaining a completion of $f_1,\ldots,f_n$ (free) through partial completions.

(3.7) Let S be a set of polynomials spanning $P_{m+1}^{\mathbf{G}}$ and let $f_1,\ldots,f_n$, $t_1,\ldots,t_k$ be a partial completion of $f_1,\ldots,f_n$ up to degree m. Let $d_i=\deg(t_i)$ and $F_i=P_i\cap R[f_1,\ldots,f_n]$. Select a maximal linearly independent subset B of S whose linear span

$[B]$ has the property $[B]\cap(\Sigma_{d_i+k=m+1} t_i F_k)=\{0\}$. Let $B=\{s_1,\ldots,s_h\}$. Then $f_1,\ldots,f_n$, $t_1,\ldots t_k$, $s_1,\ldots,s_h$ is a partial completion of $f_1,\ldots,f_n$ up to degree $m+1$.

3.8. *Essential transients*

(3.8) *If* $m>q=\max_h(H_h\neq\{0\})$, the method in (3.7) will work if the set S is replaced by the set $\{t_it_j\colon i,j=1,\ldots,k,\ \deg(t_it_j)=m+1\}$.

Because of (3.8), a set of transients in a partial completion of $f_1,\ldots,f_n$ up to degree $m>q$ is called *essential*. The remaining transients in a full completion of $f_1,\ldots,f_n$ can thus be chosen to be products of the essential transients.

3.9. *General algorithm for constructing a* GPBRI *when the representation is a sum of twisted representations of a single representation* ρ

(3.9) Let $\rho\colon \mathbf{G}\to\mathbf{O}(n)$ be a representation and $\mu_1,\ldots,\mu_k$ real linear characters of $\mathbf{G}$. Then a GPBRI for $P(\bigoplus_{i=1}^k \mu_i\rho)$ can be chosen as follows.

(a) Let $f_1,\ldots f_n$ be the free invariants for a good polynomial basis of $P(\rho)^{\mathbf{G}}$. Let C be a complement in $P(\rho)$ relative to $f_1,\ldots,f_n$. Let $\lambda_1,\ldots,\lambda_l$ be all the real linear characters of $\mathbf{G}$ and $C_{\lambda_i}=\{p\in C\colon p$ a relative invariant for $\mathbf{G}$ of weight $\lambda_i\}$, $i=1,\ldots,l$. Then there is $C'\subseteq C$ so that $C=C_{\lambda_1}\oplus\ldots\oplus C_{\lambda_l}\oplus C'$ ($\mathbf{G}$-module direct sum). The elements in the union (basis for C_{λ_l})$\cup$(basis for C_{λ_2})$\cup\ldots\cup$(basis for C_{λ_l}) form the transients of a good polynomial basis for $P(\rho)^{\hat{\mathbf{G}}}$. The free invariants of this basis are $f_1,\ldots,f_n$. This good polynomial basis for $P(\rho)^{\hat{\mathbf{G}}}$ is also a GPBRI for $P(\rho)$.

(b) Assume the notation at the beginning of § 3.6 for the representation $\bigoplus^k\rho$. For $j=1,\ldots,k$, $i=1,\ldots,n$ let $f_i^{(j)}=f_i(X_1^{(j)},\ldots,X_n^{(j)})$. Choose the free invariants for $P(\bigoplus^k\rho)^{\hat{\mathbf{G}}}$ to be the $f_i^{(j)}$s $(1\leqslant i\leqslant n,\ 1\leqslant j\leqslant k)$. We take the good polynomial basis just obtained for $P(\rho)^{\hat{\mathbf{G}}}$ in (a) and use it in (3.4) to obtain a good polynomial basis for $P(\bigoplus^{n-1}\rho)^{\hat{\mathbf{G}}}$. This will be a GPBRI for $P(\bigoplus^{n-1}\rho)$, the free invariants of which are the $f_i^{(j)}$s, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n-1$. In what follows, we will only need those transients of this basis that are of degree less than or equal to $m=q(\bigoplus^{n-1}\rho)+1$.

(Note. In using (3.4) inductively to find a GPBRI for $P(\bigoplus^{n-1}\rho)$, one must be careful to make sure that the cross-term transients for $\hat{\mathbf{G}}$ are also relative invariants for $\mathbf{G}$. In the notation of (a), C' is a $\mathbf{G}$-submodule of C. If $C^{(i)\prime}$ is C' for the ith copy of $P(\rho)$ in $P(\bigoplus^{n-1}\rho)$, then the elements of a basis for $K=(C^{(1)\prime}C^{(2)\prime}\ldots C^{(n-1)\prime})^{\hat{\mathbf{G}}}$ will be the cross-term transients. Since $C^{(1)\prime}\ldots C^{(n-1)\prime}$ is also a $\mathbf{G}$-module, K has a basis consisting of relative invariants for $\mathbf{G}$.)

(c) Let U be the set of products tf such that

(i) t is a transient of degree $\leqslant m$ obtained in (b) (including $t=t_0=1$) or $t=\det(X_i^{(j)})_{1\leqslant i,j\leqslant n}$ (if $n\leqslant m$);

(ii) f is a monomial in the $f_i^{(j)}$s, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n-1$, with $\deg f\leqslant m-\deg t$.

Then $\mathrm{Pol}(U)$ spans $(\Sigma_i^m P_i(\bigoplus^k\rho))^{\mathbf{G}}$. Use $S=\mathrm{Pol}(U)$ in (3.7) to complete $\{f_i^{(j)}\colon 1\leqslant i\leqslant n,\ 1\leqslant j\leqslant k\}$ up to degree m, thus obtaining a set of essential transients in a GPBRI for $P(\bigoplus^k\rho)$.

(d) For the basis $X_1,\ldots,X_n$ of ρ, suppose $gX_j=\Sigma_h a_{hj}(g)X_h$ for all $g\in\mathbf{G}$. Then, for each $i=1,\ldots,k$, choose a basis $Y_1^{(i)},\ldots,Y_n^{(i)}$ for the twisted representation $\mu_i\rho$ so that $gY_j^{(i)}=\Sigma_h\mu_i(g)a_{hj}(g)Y_h^{(i)}$. Let $\varphi\colon P(\bigoplus_{i=1}^k\rho)\to P(\bigoplus_{i=1}^k\mu_i\rho)$ be the unitary R-algebra homomorphism defined by $\varphi(X_j^{(i)})=Y_j^{(i)}$. Then φ(GPBRI in (c)) = GPBRI for $P(\bigoplus_{i=1}^k\mu_i\rho)$.

4. Applications to CPGs. Three important examples

In this section we will show how the ideas summarised in § 3 apply to CPGs by computing the invariants and/or harmonics for three examples important for our algorithm: (1) good polynomial bases and harmonics for $\bigoplus^m \mathbf{E}$ and $\bigoplus^m \mathbf{e}$, (2) harmonics for $\mathbf{T}_2 \oplus \mathbf{T}_2$ and $\mathbf{T} \oplus \mathbf{T}$, and (3) cross-term transients for $(\bigoplus^m \mathbf{e}) \oplus (\bigoplus^p \mathbf{T})$.

4.1. *Example 1. Good polynomial bases and harmonics for* $\bigoplus^m \mathbf{E}$ *and* $\bigoplus^m \mathbf{e}$.

From table 4 a good polynomial basis for the invariants of $\mathbf{E}$ and $\mathbf{e}$ are $\{p_1, p_2$ (free)$\}$ and $\{p_1, p_2$ (free), p_3 (transient)$\}$, respectively. To describe a good polynomial basis for $P(\bigoplus^m \mathbf{E})$ and $P(\bigoplus^m \mathbf{e}) = P(\bigoplus^m \mathbf{E})$, let u_i, v_i be a basis for the ith copy of $\mathbf{E}$ (or $\mathbf{e}$). Let $p_j^{(i)} = p_j(u_i, v_i)$, $i = 1, \ldots, m$, $j = 1, 2, 3$.

Choose the set of free elements of a good polynomial basis for $P(\bigoplus^m \mathbf{E})^{\mathbf{G}}$ to be $L = \{p_j^{(i)}: j = 1, 2, 1 \leq i \leq m\}$. Use $S = \mathrm{Pol}\{p_1, p_2\} \cup (\mathrm{Pol}\, p_1)^2$ in (3.7) to obtain a partial completion of L up to degree 4. The transients obtained will be a set of essential transients of a good polynomial basis. This procedure uses a fact from Weyl (1946, p 37) and is a variation on (3.9)(c).

Similarly, choose the set of free elements of a good polynomial basis for $P(\bigoplus^m \mathbf{e})^{\mathbf{G}}$ to be $\{p_j^{(i)}: j = 1, 2, 3, 1 \leq i \leq m\}$. Let $\alpha = u_1 v_2 - v_1 u_2$ and proceed as in (3.9)(c) to find the essential transients.

Let $H(m)$ and $H'(m)$ denote the harmonic polynomials in $P(\bigoplus^m \mathbf{E})$ and $P(\bigoplus^m \mathbf{e})$, respectively. Then, using (3.6) plus methods in the proof of (3.6) together with the fact from Weyl, we have

(4.1) (a) $H_1(m) = \mathrm{Pol}(H_1(\mathbf{E}))$ with $\mathbf{G}$-action $\mathbf{E} \oplus \ldots \oplus \mathbf{E}$.
(b) $H_1'(m) = \mathrm{Pol}(H_1(\mathbf{e}))$ with $\hat{\mathbf{G}}$-action $\mathbf{e} \oplus \ldots \oplus \mathbf{e}$.
(c) $H_2(m) = \mathrm{Pol}(H_2(\mathbf{E}))$ with $\mathbf{G}$-action $\mathbf{E} \oplus \ldots \oplus \mathbf{E}$.
(d) $H_2'(m) = \mathrm{Pol}(H_2(\mathbf{e}))$ with $\hat{\mathbf{G}}$-action $\mathbf{e} \oplus \ldots \oplus \mathbf{e}$.
(e) $H_3(m) = \mathrm{Pol}(H_3(\mathbf{E}))$ with $\mathbf{G}$-action $\mathbf{B} \oplus \ldots \oplus \mathbf{B}$.
(f) $H_3'(m) = \{0\} = H_4(m)$.

4.2. *Example 2. Harmonics for* $\mathbf{T}_2 \oplus \mathbf{T}_2$ *and* $\mathbf{T} \oplus \mathbf{T}$.

The non-identity irreducible representations of $\mathbf{T}_2$ are $\mathbf{B}$, $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{E}$. Those of $\mathbf{T}$ are $\mathbf{T}$ and $\mathbf{e}$. To describe H, the harmonics of $\mathbf{T}_2 \oplus \mathbf{T}_2$, and H', the harmonics of $\mathbf{T} \oplus \mathbf{T}$, we need to adopt some notation.

Let $P_{r_1 r_2} \subseteq P = P(\mathbf{T}_2 \oplus \mathbf{T}_2)$ denote those polynomials that are homogeneous of degree r_1 in X_1, Y_1, Z_1 (first copy of $\mathbf{T}_2$) and homogeneous of degree r_2 in X_2, Y_2, Z_2 (second copy of $\mathbf{T}_2$). Thus, for all k, $P_k = \bigoplus_{r_1 + r_2 = k} P_{r_1 r_2}$. Furthermore, let $H_{r_1 r_2} = H \cap P_{r_1 r_2}$, $H'_{r_1 r_2} = H' \cap P_{r_1 r_2}$. Since $P_{r_1 r_2}$, H and H' are invariant under $\mathbf{G}$ so are $H_{r_1 r_2}$ and $H'_{r_1 r_2}$. Also $H_k = \bigoplus_{r_1 + r_2 = k} H_{r_1 r_2}$ and $H'_k = \bigoplus_{r_1 + r_2 = k} H'_{r_1 r_2}$. Table 6 displays the action of $\mathbf{G}$ on $H_{r_1 r_2}$ and $H'_{r_1 r_2}$ by giving the multiplicities of $\mathbf{B}$, $\mathbf{E}$, $\mathbf{T}_1$ and $\mathbf{T}_2$ in each. The action of $\mathbf{T} = \hat{\mathbf{T}}_2$ on $\mathbf{E}$ is $\mathbf{e}$ and on $\mathbf{T}_1$ and $\mathbf{T}_2$ is $\mathbf{T}$.

4.3. *Example 3. The cross-term transients of a good polynomial basis for the invariants of* $(\bigoplus^m \mathbf{e}) \oplus (\bigoplus^p \mathbf{T})$.

Having found good polynomial bases for the invariants of $\bigoplus^m \mathbf{e}$ and $\bigoplus^p \mathbf{T}$, we can use (3.4) to describe a good polynomial basis for the sum of these representations. The

Table 6. Action of T_2 on the harmonic polynomials H of $T_2 \oplus T_2$ and on the harmonic polynomials H' of $T \oplus T = \widehat{T_2 \oplus T_2}$.

Polynomial degree k	Homogeneous bi-degree $r_1 r_2$	T_2-action by multiplicities on H				on H'		
		B	**E**	T_1	T_2	**E**	T_1	T_2
1	1 0	0	0	0	1	0	0	1
	0 1	0	0	0	1	0	0	1
2	2 0	0	1	0	1	1	0	1
	1 1	0	1	1	1	1	1	1
	0 2	0	1	0	1	1	0	1
3	3 0	0	0	1	1	0	1	1
	2 1	0	1	2	2	1	2	2
	1 2	0	1	2	2	1	2	2
	0 3	0	0	1	1	0	1	1
4	4 0	0	1	1	0	1	1	0
	3 1	1	1	2	0	1	2	0
	2 2	1	1	2	0	1	2	0
	1 3	1	1	2	0	1	2	0
	0 4	0	1	1	0	1	1	0
5	5 0	0	0	1	0	0	1	0
	4 1	1	0	0	1	0	0	1†
	3 2	1	0	0	0	0	0	0
	2 3	1	0	0	0	0	0	0
	1 4	1	0	0	1	0	0	1†
	0 5	0	0	1	0	0	1	0
6	6 0	1	0	0	0	0	0	0
	5 1	1	0	0	0	0	0	0
	4 2	1	0	0	0	0	0	0
	3 3	1	0	0	0	0	0	0
	2 4	1	0	0	0	0	0	0
	1 5	1	0	0	0	0	0	0
	0 6	1	0	0	0	0	0	0

† Except for the lack of subspaces in H' with T_2-action **B**, this is the first place where H' could differ from H by virtue of the fact that relative invariants for T_2 that are not invariants first appear in degree 4. However, the only candidate for a piece of $P_{4\,1}$ to appear in the invariant ideal for **T** but not in the invariant ideal for T_2 is the space $SH_{1\,0}$. Here S is the subspace of $H_{3\,1}$ on which T_2 acts like **B**. But the action of T_2 on $SH_{1\,0}$ is like T_2 so that $SH_{1\,0}$ is in the invariant ideal for T_2 as well as for **T**.

interesting part of (3.4) is to determine the cross-term transients of the good polynomial basis. We will use the method outlined in the remark following theorem 3 of Gay and Ascher (1984) to describe the cross-term transients.

Let $H(\mathbf{e}, m)$ and $H(\mathbf{T}, p)$ denote the harmonics of $\oplus^m \mathbf{e}$ and $\oplus^p \mathbf{T}$, respectively. Write $H(\mathbf{e}) = H(\mathbf{e}, 1)$ and $H(\mathbf{T}\oplus\mathbf{T}) = H(\mathbf{T}, 2)$. From (3.7), $H(\mathbf{e}, m) = \mathrm{Pol}(H(\mathbf{e}))$ and $H(\mathbf{T}, p) = \mathrm{Pol}(H(\mathbf{T}\oplus\mathbf{T}))$. In particular, irreducible summands for $H(\mathbf{e}, m)$ can be chosen to be of the form $\partial(A)$ where A is an irreducible summand of $H(\mathbf{e})$ and ∂ is an appropriate product of polarisations. Similarly an irreducible summand for $H(\mathbf{T}, p)$ can be chosen to be of the form $\partial'(B)$ where B is an irreducible summand of $H(\mathbf{T}\oplus\mathbf{T})$ and ∂' is an appropriate product of polarisations. Thus a typical basis element for

$(H(\mathbf{e}, m)H(\mathbf{T}, p))^{\mathbf{G}}$ can be chosen to be contained in a $(\partial(A)\partial'(B))^{\mathbf{G}}$. Now we know that $(\partial(A)\partial'(B))^{\mathbf{G}} \neq \{0\}$ iff $\partial(A) \neq \{0\} \neq \partial'(B)$ and the action of **G** on $\partial(A)$ and $\partial'(B)$ are the same. Furthermore, if $(\partial(A)\partial'(B))^{\mathbf{G}} \neq \{0\}$ then also $(AB)^{\mathbf{G}} \neq \{0\}$ and $(AB)^{\mathbf{G}}$ is spanned by a single invariant p in case A is absolutely irreducible and by two linearly independent invariants p, q in case A is not absolutely irreducible. It is not difficult to show that $(\partial A \partial B)^{\mathbf{G}}$ is then spanned by $\partial\partial'(p)$ in the first case and by $\partial\partial'(p)$ and $\partial\partial'(q)$ in the second.

Thus cross-term transients for $(\bigoplus^m \mathbf{e}) \oplus (\bigoplus^p \mathbf{T})$ can be obtained by applying certain polar operators to the cross-term transients of $\mathbf{e} \oplus (\mathbf{T} \oplus \mathbf{T})$. The only irreducible constituents of $H(\mathbf{e})$ and $H(\mathbf{T} \oplus \mathbf{T})$ on which the **G**-action is the same are those where the **G**-action is **e**. For $H(\mathbf{e})$ these are $H_1(\mathbf{e})$ and $H_2(\mathbf{e})$, the homogeneous harmonics of degree 1 and 2, respectively. For $H(\mathbf{T} \oplus \mathbf{T})$ these are K_{ij} where ij is a bi-degree in the set

$$I = \{20, 11, 02, 21, 12, 40, 31, 22, 13, 04\}$$

and K_{ij} is the unique irreducible constituent in $H(\mathbf{T} \oplus \mathbf{T})$ of bi-degree ij on which **G** acts like **e**. Furthermore, **e** is irreducible but not absolutely irreducible.

Consequently the cross-term transients can be described as follows.

(4.3) For $i = 1, 2$ and $jk \in I$, let p_{ijk} denote the pair a, b of linearly independent polynomials spanning $(H_i(\mathbf{e})K_{jk})^{\mathbf{G}}$. Let ∂ be a product of polarisations from **e** to $\bigoplus^m \mathbf{e}$ and let ∂' be a product of polarisations from $\mathbf{T} \oplus \mathbf{T}$ to $\bigoplus^p \mathbf{T}$. Then the cross-term transients for the good polynomial basis of $P((\bigoplus^m \mathbf{e}) \oplus (\bigoplus^p \mathbf{T}))^{\mathbf{G}}$ are spanned by pairs of polynomials of the form $\partial\partial'(a)$, $\partial\partial'(b)$. The pairs p_{ijk} are listed in table 7.

5. An algorithm for finding a good polynomial basis of relative invariants for CPGS

In this section we will describe a method for obtaining a good polynomial basis for the relative invariants (GPBRI), in the sense of § 1, associated with any representation of a CPG. From what we have seen (theorem 1, tables 4 and 5), it is sufficient to find GPBRIS for the following representations:

(1) $\bigoplus^{r} \mu_i$

(2) $\bigoplus^{r} \mu_i + \bigoplus^{s} \nu_j \mathbf{F}$

(3) $\bigoplus^{r} \mu_i + \bigoplus^{s} \nu_j \mathbf{E} + \bigoplus^{t} \lambda_k \mathbf{T}_2$

where the μ_is, ν_js and λ_ks are one-dimensional representations.

By (3.9) and Theorem 13 of Gay and Ascher (1984) we will know GPBRIS for all these once we know GPBRIS for

(a) $\bigoplus^{s} \mathbf{F}$,

(b) $\bigoplus^{s} \mathbf{E}$,

(c) $\bigoplus^{t} \mathbf{T}_2$,

(d) $\bigoplus^{s} \mathbf{E} + \bigoplus^{t} \mathbf{T}_2$.

This is what we do below. In each case we will describe the free invariants and a set of essential transients of the GPBRI.

Table 7. The cross term transient pairs of $P(\sigma \oplus \rho)$ where $\sigma(\mathbf{G}) = \mathbf{e}$ and $\rho(\mathbf{G}) = \mathbf{T} \oplus \mathbf{T}$ as described in (4.3). See also (3.4). The variables of **e** are u, v; the variables of $\mathbf{T} \oplus \mathbf{T}$ are x_1, y_1, z_1 (first copy), x_2, y_2, z_2 (second copy).

Cross term transient pairs†	p_{ijk} (in notation of (4.3)
$u(2z_1^2 - x_1^2 - y_1^2) + \sqrt{3}v(x_1^2 - y_1^2).$	p_{120}
$u(2z_2^2 - x_2^2 - y_2^2) + \sqrt{3}v(x_2^2 - y_2^2).$	p_{102}
$(u^2 - v^2)(2z_1^2 - x_1^2 - y_1^2) - 2\sqrt{3}uv(x_1^2 - y_1^2).$	p_{220}
$(u^2 - v^2)(2z_2^2 - x_2^2 - y_2^2) - 2\sqrt{3}uv(x_2^2 - y_2^2).$	p_{202}
$u(2z_1 z_2 - x_1 x_2 - y_1 y_2) + \sqrt{3}v(x_1 x_2 - y_1 y_2).$	p_{111}
$(u^2 - v^2)(2z_1 z_2 - x_1 x_2 - y_1 y_2) - 2\sqrt{3}uv(x_1 x_2 - y_1 y_2).$	p_{211}
$u(2x_1 y_1 z_2 - y_1 z_1 x_2 - z_1 x_1 y_2) + \sqrt{3}v(y_1 z_1 x_2 - z_1 x_1 y_2).$	p_{121}
$u(2x_2 y_2 z_1 - y_2 z_2 x_1 - z_2 x_2 y_1) + \sqrt{3}v(y_2 z_2 x_1 - z_2 x_2 y_1).$	p_{112}
$(u^2 - v^2)(2x_1 y_1 z_2 - y_1 z_1 x_2 - z_1 x_1 y_2) - 2\sqrt{3}uv(y_1 z_1 x_2 - z_1 x_1 y_2).$	p_{221}
$(u^2 - v^2)(2x_2 y_2 z_1 - y_2 z_2 x_1 - z_2 x_2 y_1) - 2\sqrt{3}uv(y_2 z_2 x_1 - z_2 x_2 y_1).$	p_{212}
$u[6(2x_1^2 y_1^2 - y_1^2 z_1^2 - z_1^2 x_1^2) + 2z_1^4 - x_1^4 - y_1^4] - \sqrt{3}v[6(x_1^2 - y_1^2)z_1^2 + y_1^4 - x_1^4].$	p_{140}
$u[6(2x_2^2 y_2^2 - y_2^2 z_2^2 - z_2^2 x_2^2) + 2z_2^4 - x_2^4 - y_2^4] - \sqrt{3}v[6(x_2^2 - y_2^2)z_2^2 + y_2^4 - x_2^4].$	p_{104}
$(u^2 - v^2)[6(2x_1^2 y_1^2 - y_1^2 z_1^2 - z_1^2 x_1^2) + 2z_1^4 - x_1^4 - y_1^4]$ $+ 2\sqrt{3}uv[6(x_1^2 - y_1^2)z_1^2 + y_1^4 - x_1^4].$	p_{240}
$(u^2 - v^2)[6(2x_2^2 y_2^2 - y_2^2 z_2^2 - z_2^2 x_2^2) + 2z_2^4 - x_2^4 - y_2^4]2\sqrt{3}uv[6(x_2^2 - y_2^2)z_2^2 + y_2^4 - x_2^4].$	p_{204}
$3u[2x_1 y_1(x_1 y_2 + y_1 x_2) - y_1 z_1(y_1 z_2 + z_1 y_2) - z_1 x_1(z_1 x_2 + x_1 z_2)]$ $+ u(2z_1^3 z_2 - x_1^3 x_2 - y_1^3 y_2) + 3\sqrt{3}v[y_1 z_1(y_1 z_2 + z_1 y_2)$ $- z_1 x_1(z_1 x_2 + x_1 z_2)] + \sqrt{3}v(x_1^3 x_2 - y_1^3 y_2).$	p_{131}
$3u[2x_2 y_2(x_2 y_1 + y_2 x_1) - y_2 z_2(y_2 z_1 + z_2 y_1) - z_2 x_2(z_2 x_1 + x_2 z_1)]$ $+ u(2z_2^3 z_1 - x_2^3 x_1 - y_2^3 y_1) + 3\sqrt{3}v[y_2 z_2(y_2 z_1 + z_2 y_1)$ $- z_2 x_2(x_2 x_1 + x_2 z_1)] + \sqrt{3}v(x_2^3 x_1 - y_2^3 y_1).$	p_{113}
$3(u_2 - v^2)[2x_1 y_1(x_1 y_2 + y_1 x_2) - y_1 z_1(y_1 z_2 + z_1 y_2) - z_1 x_1(z_1 x_2 + x_1 z_2)]$ $+ (u^2 - v^2)(2z_1^3 z_2 - x_1^3 x_2 - y_1^3 y_2)$ $- 6\sqrt{3}uv[y_1 z_1(y_1 z_2 + z_1 y_2) - z_1 x_1(z_1 x_2 + x_1 z_2)] - 2\sqrt{3}uv(x_1^3 x_2 - y_1^3 y_2).$	p_{231}
$3(u^2 - v^2)[2x_2 y_2(x_2 y_1 + y_2 x_1) - y_2 z_2(y_2 z_1 + z_2 y_1) - z_2 x_2(z_2 x_1 + x_2 z_1)]$ $+ (u_2 - v^2)(2z_2^3 z_1 - x_2^3 x_1 - y_2^3 y_1)$ $- 6\sqrt{3}uv[y_2 z_2(y_2 z_1 + z_2 y_1) - z_2 x_2(z_2 x_1 + x_2 z_1)] - 2\sqrt{3}uv(x_2^3 x_1 - y_2^3 y_1).$	p_{213}
$4u(2x_1 y_1 x_2 y_2 - y_1 z_1 y_2 z_2 - z_1 x_1 z_2 x_2)$ $+ u[x_1^2(2y_2^2 - z_2^2 - x_2^2) + y_1^2(2x_2^2 - y_2^2 - z_2^2) + z_1^2(2z_2^2 - x_2^2 - y_2^2)]$ $- 4\sqrt{3}v(y_1 z_1 y_2 z_2 - z_1 x_1 z_2 x_2)$ $+ \sqrt{3}v[x_1^2(z_2^2 - x_2^2) + y_1^2(y_2^2 - z_2^2) + z_1^2(x_2^2 - y_2^2)].$	p_{122}
$4(u^2 - v^2)(2x_1 y_1 x_2 y_2 - y_1 z_1 y_2 z_2 - z_1 x_1 z_2 x_2)$ $+ (u^2 - v^2)[x_1^2(2y_2^2 - z_2^2 - x_2^2) + y_1^2(2x_2^2 - y_2^2 - z_2^2)$ $+ z_1^2(2z_2^2 - x_2^2 - y_2^2)] + 8\sqrt{3}uv(y_1 z_1 y_2 z_2 - z_1 x_1 z_2 x_2)$ $- 2\sqrt{3}uv[x_1^2(z_2^2 - x_2^2) + y_1^2(y_2^2 - z_2^2) + z_1^2(x_2^2 - y_2^2)].$	p_{222}

† Only one polynomial of each pair is given. In case the pair is p_{1ij}, the remaining polynomial in the pair can be obtained by replacing u by v and v by $-u$. In case the pair is p_{2ij}, the other polynomial can be obtained by replacing $u^2 - v^2$ by $2uv$ and $2uv$ by $v^2 - u^2$.

5.1. Representation of the form $\bigoplus^s$ **F**.

From table 4, a GPBRI for **F** is

$$\mathscr{B} = \{p^2+q^2, p^2-q^2(\text{free}), pq(\text{transient})\}.$$

Choose the free part of a GPBRI for $\bigoplus^s \mathbf{F}$ to be $L=\{p_i^2+q_i^2, p_i^2-q_i^2\colon i=1,\ldots,s\}$.

Apply polar operators from **F** to $\bigoplus^s \mathbf{F}$ on $\mathscr{B}$. In (3.7) use $S=\mathrm{Pol}(\mathscr{B})$ to partially complete L up to degree 2. This will yield a set of essential transients for the GPBRI (of degree $\leqslant 2$) by (3.9)(c).

5.2. Representation of the form $\bigoplus^s$ **E**.

This was done in § 4.1.

5.3. Representation of the form $\bigoplus^t$ $\mathbf{T}_2$.

From table 4, the free part of a GPBRI for $\mathbf{T}_2 \oplus \mathbf{T}_2$ is

$$\{x_i^2+y_i^2+z_i^2, x_iy_iz_i, x_i^4+y_i^4+z_i^4\colon i=1,2\}.$$

The essential transients for this GPBRI are

$$\{(x_i^2-y_i^2)(y_i^2-z_i^2)(z_i^2-x_i^2)\colon i=1,2\}$$

plus the 28 cross-term transients of degree $\leqslant 6$ in table 8. Let $\mathscr{B}$ denote the set of 6 free and 30 essential transient elements of the GPBRI for $\mathbf{T}_2 \oplus \mathbf{T}_2$ together with

$$\det\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}.$$

Table 8. Cross-term transients of degree $\leqslant 6$ for $\mathbf{T}\oplus\mathbf{T}$.

Bi-Degree†	
11	$x_1x_2+\ldots$
21	$x_1y_1z_2+\ldots$
31	$(y_1^2+z_1^2)x_1x_2+\ldots$
31	$(y_1^2-z_1^2)x_1x_2+\ldots$
22	$x_1y_1x_2y_2+\ldots$
22	$(2z_1^2-x_1^2-y_1^2)(2z_2^2-x_2^2-y_2^2)+3(x_1^2-y_1^2)(x_2^2-y_2^2)$
22	$(2z_1^2-x_1^2-y_1^2)(x_2^2-y_2^2)-(x_1^2-y_1^2)(2z_2^2-x_2^2-y_2^2)$
41	$(y_1^2-z_1^2)y_1z_1x_2+\ldots$
32	$(y_1^2-z_1^2)x_1y_2z_2+\ldots$
60	$(x_1^2-y_1^2)(y_1^2-z_1^2)(z_1^2-x_1^2)$
51	$(y_1^2-z_1^2)x_1^3x_2+\ldots$
42	$(y_1^2-z_1^2)y_1z_1y_2z_2+\ldots$
42	$(2z_1^4-x_1^4-y_1^4)(2z_2^4-x_2^4-y_2^4)+3(x_1^4-y_1^4)(x_2^4-y_2^4)$
42	$(2z_1^4-x_1^4-y_1^4)(x_2^4-y_2^4)-(x_1^4-y_1^4)(2z_2^2-x_2^2-y_2^2)$
33	$(y_1^2-z_1^2)x_1x_2(y_2^2-z_2^2)+\ldots$
33	$(y_1^2-z_1^2)x_1x_2(y_2^2+z_2^2)+\ldots$
33	$(y_1^2+z_1^2)x_1x_2(y_2^2+z_2^2)+\ldots$

† For every cross-term transient of bi-degree ij there is a corresponding cross-term transient of bi-degree ji obtained by interchanging x_1, y_1, z_1 with x_2, y_2, z_2. When $i=j$, the result of this interchange may be the same (but not always).

Choose the free part of a GPBRI for $P(\bigoplus^t \mathbf{T}_2)$ to be

$$L = \{x_i^2 + y_i^2 + z_i^2,\ x_i y_i z_i,\ x_i^4 + y_i^4 + z_i^4,\ i = 1, \ldots, t\}$$

Apply the polar process to $\mathscr{B}$ and use $S = \mathrm{Pol}(\mathscr{B})$ in (3.7) to partially complete L up to degree 6. This will yield a set of essential transients (of degree ≤ 6) for the GPBRI for $P(\bigoplus^t \mathbf{T}_2)$.

5.4. Representations of the form $(\bigoplus^s \boldsymbol{E}) \oplus (\bigoplus^t \boldsymbol{T}_2)$.

The free part of the GPBRI is the union of the free parts for $\bigoplus^s \mathbf{E}$ and $\bigoplus^t \mathbf{T}_2$ as obtained in §§ 5.2 and 5.3.

The set of transients of the GPBRI is the union of the following sets.

(1) The set of transients of the GPBRI for $\bigoplus^s \mathbf{E}$ obtained from the essential transients in § 5.2 using (3.8).

(2) The set of transients of the GPBRI for $\bigoplus^t \mathbf{T}_2$ as obtained from the essential transients in § 5.3 using (3.8).

(3) The set of products pq where p is from set (1) and q is from set (2).

(4) A maximal linearly independent subset of S where S is the set of polynomials $\partial\partial'(p)$ where ∂ is a polar operator from $\mathbf{E}$ to $\bigoplus^s \mathbf{E}$, ∂' is a polar operator from $\mathbf{T}_2 \oplus \mathbf{T}_2$ to $\bigoplus^t \mathbf{T}_2$ and p is a relative invariant of $\mathbf{E} \oplus \mathbf{T}_2 \oplus \mathbf{T}_2$ listed in table 7.

6. Appendix. Methods used to construct tables 6, 7 and 8

In this section we will describe the computations which resulted in tables 6, 7 and 8.

Many of the tools we use are not new. For example, the Clebsch–Gordon coefficients and the polynomials in table 9 are known. (See, for instance, Kopský (1976).) The method we will use below in case the matrix group is $\mathbf{T}_2 \oplus \mathbf{T}_2$ has been sketched in general in Kopský (1979a). For completeness and ease of understanding, we have included most of the tools necessary for its execution.

The method is recursive: in order to derive facts about homogeneous polynomials of degree m, we must first know all pertinant facts about P_k for all $k < m$.

Let $P = P(\mathbf{T}_2 \oplus \mathbf{T}_2)$ and let Q denote a complement relative to $P^{\mathbf{G}}$. Let P_{pq} denote the homogeneous polynomials of degree p in the variables associated with the first copy of T_2 and degree q in the variables associated with the second copy. Since P_{pq} is a $\mathbf{G}$-submodule of P so also is $Q_{pq} = Q \cap P_{pq}$. Our job is to decompose each Q_{pq} into $\mathbf{G}$-irreducible subspaces, to find (in some cases) bases for these subspaces and to characterise how $\mathbf{G}$ acts on them. At the same time we want to find a basis for $P_{pq}^{\mathbf{G}}$.

Step 1. The irreducible action of $\mathbf{G}$ occurring in a decomposition of P_{pq} will be one of $\mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{T}_1$ and $\mathbf{T}_2$. If

$$P_{pq} \simeq \underbrace{\mathbf{A} \oplus \ldots \oplus \mathbf{A}}_{n_1} \oplus \underbrace{\mathbf{B} \oplus \ldots \oplus \mathbf{B}}_{n_2} \oplus \ldots \oplus \underbrace{\mathbf{T}_2 \oplus \ldots \oplus \mathbf{T}_2}_{n_5}$$

then we call $n_1, n_2, \ldots, n_5$ the *irreducible multiplicities* for the action of $\mathbf{G}$ in P_{pq}. To determine these multiplicites for P_{pq}, we use these facts

(a) $P_{pq} = P_{p0} P_{0q}$

(b) $P_{p0} = \bigoplus_{\mu=0}^{p} P_{p-\mu,0}^{\mathbf{G}} Q_{\mu 0}$ and $P_{0q} = \bigoplus_{\nu=0}^{q} P_{0,q-\nu}^{\mathbf{G}} Q_{0\nu}$

(c) The irreducible multiplicities for $Q_{\mu 0}$ and $Q_{0\nu}$ can be obtained from table 4.

(d) If the irreducible multiplicities for $Q_{\mu 0}$ are $a_1, a_2, \ldots, a_5$ and $b = \dim P^{\mathbf{G}}_{p-\mu,0}$ then the irreducible multiplicities for $P^{\mathbf{G}}_{p-\mu,0}Q_{\mu 0}$ are $a_1 b, a_2 b, \ldots, a_5 b$.

(e) Dim $P^{\mathbf{G}}_{p-\mu,0}$ can be found easily from table 4 using the fact that $\mathbf{T}_2$ is a reflection group.

The multiplicities for P_{p0} and P_{0q} are then easily found using (b), (c), (d) and (e). From these, the irreducible multiplicities for P_{pq} are then determined using table 9 and the fact that $P_{pq} = P_{p0}P_{0q} \simeq P_{p0} \otimes P_{0q}$.

Step 2. (a) In table 4, bases for each irreducible constituent of $Q_{\mu 0}$ and $Q_{0\nu}$ are given.

(b) From table 4 and the fact that $\mathbf{T}_2$ is a reflection group find bases for $P^{\mathbf{G}}_{\mu 0}$ and $P^{\mathbf{G}}_{0\nu}$ $(2 \leqslant \mu, \nu \leqslant 6)$. (This may be done simultaneously with step 1(c).)

Step 3. (Recursive Step). Fix positive integers $p \neq 0$ and $q \neq 0$. Assume that bases $\mathscr{B}_{\mu\nu}$ for $P^{\mathbf{G}}_{\mu\nu}$ have been found for all pairs (μ, ν) with $\mu \leqslant p$, $\nu \leqslant q$ and $(\mu, \nu) \neq (p, q)$. For all such (μ, ν)'s, assume also that a basis for each irreducible constituent of $Q_{\mu\nu}$ has been found (relative to some decomposition of $Q_{\mu\nu}$ into irreducible subspaces).

Step 4. Find a maximal linearly independent subset $\mathscr{L}_{pq}$ in $\bigcup \{\mathscr{B}_{\mu_1\nu_1} \ldots \mathscr{B}_{\mu_k\nu_k}: \Sigma_i\, \mu_i = p, \Sigma_i\, \nu_i = q\}$.

Step 5. To find a decomposition of $Q_{p0}Q_{0q}$ into irreducible constituents and find a basis for each, first decompose Q_{p0} and Q_{0q} into irreducible constituents:

$$Q_{p0} = C_1 \oplus \ldots \oplus C_k, \qquad Q_{0q} = D_1 \oplus \ldots \oplus D_l.$$

A basis for each C_i and D_i will have already been recorded in step 2(a) above (from table 4). Next we want to decompose each C_iD_j into irreducible constituents, find a basis for each and how $\mathbf{G}$ acts on it. Table 9 shows how to do this given bases for C_i and D_j and how the group acts on C_i and D_j.

Table 9. Decomposition of the tensor product of two irreducible representations of $\mathbf{s}_4$ into irreducible constituents. A basis for each constituent is given in terms of bases for the two factors.

$\otimes$	$\mathbf{E}$: $[u, v]$	$\mathbf{T}_1$: $[r, s, t]$	$\mathbf{T}_2$: $[x, y, z]$
$\mathbf{E}$	$\mathbf{A}$: $u_1u_2 + v_1v_2$ $\mathbf{B}$: $u_1v_2 - v_1u_2$ $\mathbf{E}$: $[u_1u_2 - v_1v_2, -(u_1v_2 + v_1u_2)]$	$\mathbf{T}_1$: $[(u - \sqrt{3}v)r, (u + \sqrt{3}v)s, -2ut]$ $\mathbf{T}_2$: $[(\sqrt{3}u + v)r, (-\sqrt{3}u + v)s, -2vt]$	$\mathbf{T}_1$: $[(\sqrt{3}u + v)x, -(\sqrt{3}u + v)y, -2vz]$ $\mathbf{T}_2$: $[(u - \sqrt{3}v)x, (u + \sqrt{3}v)y, -2uz]$
$\mathbf{T}_1$		$\mathbf{A}$: $r_1r_2 + s_1s_2 + t_1t_2$ $\mathbf{E}$: $[2t_1t_2 - r_1r_2 - s_1s_2, \sqrt{3}(r_1r_2 - s_1s_2)]$ $\mathbf{T}_1$: $[s_1t_2 - t_1s_2, \ldots]$ $\mathbf{T}_2$: $[s_1t_2 + t_1s_2, \ldots]$	$\mathbf{B}$: $rx + sy + tz$ $\mathbf{E}$: $[-\sqrt{3}(rx - sy), 2tz - rx - sy]$ $\mathbf{T}_1$: $[sz + ty, \ldots]$ $\mathbf{T}_2$: $[sz - ty, \ldots]$
$\mathbf{T}_2$			$\mathbf{A}$: $x_1x_2 + y_1y_2 + z_1z_2$ $\mathbf{E}$: $[2z_1z_2 - x_1x_2 - y_1y_2, \sqrt{3}(x_1x_2 - y_1y_2)]$ $\mathbf{T}_1$: $[y_1z_2 - z_1y_2, \ldots]$ $\mathbf{T}_2$: $[y_1z_2 + z_1y_2, \ldots]$

For an irreducible representation Γ of $\mathbf{G}$, let $\mathscr{B}(\Gamma, p, q)$ be a basis for the $\acute{\Gamma}$-part of $Q_{p0}Q_{0q}$, i.e., the subspace of $Q_{p0}Q_{0q}$ on which $\mathbf{G}$ acts like Γ.

Step 6. From $\mathscr{B}(\boldsymbol{A}, p, q) \cup \mathscr{L}_{pq}$ extract a maximal linearly independent subset $\mathscr{B}_{pq}$, a basis for $P_{pq}^{\mathbf{G}}$. (This follows from (3.2).)

Table 10. $\mathbf{T}_2 \oplus \mathbf{T}_2$. Good polynomial basis and basis for a complement. (Only constituents where the group acts like **B** or **E** are considered. Invariants appear under Irrep **A**.)

Homogeneous bi-degree	Irrep Γ	Polynomial Basis
20	**A**	$x_1^2+y_1^2+z_1^2$ (free)
	B	$\varnothing$
	E	$\begin{bmatrix} 2z_1^2-x_1^2-y_1^2 \\ \sqrt{3}(x_1^2-y_1^2) \end{bmatrix}$
11	**A**	$x_1x_2+y_1y_2+z_1z_2$ (transient)
	B	$\varnothing$
	E	$\begin{bmatrix} 2z_1z_2-x_1x_2-y_1y_2 \\ \sqrt{3}(x_1x_2-y_1y_2) \end{bmatrix}$
30	**A**	$x_1y_1z_1$ (free)
21	**A**	$x_1y_1z_2+y_1z_1x_2+z_1x_1y_2$ (transient)
	B	$\varnothing$
	E	$\begin{bmatrix} 2x_1y_1z_2-y_1z_1x_2-z_1x_1y_2 \\ \sqrt{3}(y_1z_1x_2-z_1x_1y_2) \end{bmatrix}$
40	**A**	$x_1^4+y_1^4+z_1^4$ (free)
	B	$\varnothing$
	E	$\begin{bmatrix} 6(2x_1^2y_1^2-y_1^2z_1^2-z_1^2x_1^2)+2z_1^4-x_1^4-y_1^4 \\ \sqrt{3}[6z_1^2(y_1^2-x_1^2)+x_1^4-y_1^4] \end{bmatrix}$
31	**A**	$(y_1^2+z_1^2)x_1x_2+\dots$ (transient)
	B	$(y_1^2-z_1^2)x_1x_2+(z_1^2-x_1^2)y_1y_2+\dots$
	E	$3\begin{bmatrix} 2x_1y_1(x_1y_2+y_1x_2)-y_1z_1(y_1z_2+z_1y_2)-z_1x_1(z_1x_2+x_1z_2) \\ \sqrt{3}\{y_1z_1(y_1z_2+z_1y_2)-z_1x_1(z_1x_2+x_1z_2)\} \end{bmatrix} + \begin{bmatrix} 2z_1^3z_2-x_1^3x_2-y_1^3y_2 \\ \sqrt{3}(x_1^3x_2-y_1^3y_2) \end{bmatrix}$
22	**A**	$x_1y_1x_2y_2+\dots$ (transient)
	B	$(2z_1^2-x_1^2-y_1^2)(x_2^2-y_2^2)-(x_1^2-y_1^2)(2z_2^2-x_2^2-y_2^2)$
	E	$4\begin{bmatrix} 2x_1y_1x_2y_2-y_1z_1y_2z_2-z_1x_1z_2x_2 \\ -\sqrt{3}(y_1z_1y_2z_2-z_1x_1z_2x_2) \end{bmatrix} + \begin{bmatrix} x_1^2(2y_2^2-z_2^2-x_2^2)+y_1^2(2x_2^2-y_2^2-z_2^2)+z_1^2(2z_2^2-x_2^2-y_2^2) \\ \sqrt{3}\{x_1^2(z_2^2-x_2^2)+y_1^2(y_2^2-z_2^2)+z_1^2(x_2^2-y_2^2)\} \end{bmatrix}$
41	**B**	$(y_1^2-z_1^2)y_1z_1x_2+\dots$
32	**B**	$(y_1^2-z_1^2)x_1y_2z_2+\dots$
60	**B**	$(x_1^2-y_1^2)(y_1^2-z_1^2)(z_1^2-x_1^2)$
51	**B**	$(y_1^2-z_1^2)x_1^3x_2+\dots$
42	**B**	$(y_1^2-z_1^2)y_1z_1y_2z_2+\dots$
33	**B**	$(y_1^2-z_1^2)x_1x_2(y_2^2-z_2^2)+\dots$

Step 7. (a) For $\mu \leqslant p$, $\nu \leqslant q$ and $(\mu, \nu) \neq (p, q)$, let $\Gamma_{\mu\nu}$ denote the Γ-part of $Q_{\mu\nu}$. We want to find the Γ-part of $I_{pq} = P_{pq} \cap I$. In symbols, this is equal to

$$(*) \qquad \sum_{\substack{\mu \leqslant p, \nu \leqslant q \\ (\mu,\nu) \neq (p,q)}} P^{\mathbf{G}}_{p-\mu, q-\nu} \Gamma_{\mu\nu}.$$

Since this sum is not necessarily direct, we will have to work at finding a basis for the Γ-part of I_{pq}. In step 3, we already computed bases for $P^{\mathbf{G}}_{p-\mu,q-\nu}$ and $\Gamma_{\mu\nu}$. Thus it is easy to find a basis for each $P^{\mathbf{G}}_{p-\mu,q-\nu}\Gamma_{\mu\nu}$ (it is equal to the set of products!). The union of the bases for all summands of (*) is, of course, a spanning set for the Γ-part of I_{pq}. We use Gaussian elimination to extract a basis $C(\Gamma, p, q)$ from this spanning set. From there is it is easy to determine the multiplicity b of Γ in I_{pq}.

(b) Let a denote the multiplicity of Γ in P_{pq} obtained in step 1. Then $a-b$ is the multiplicity of Γ in Q_{pq}. (Warning: $a-b$ is not necessarily the multiplicity of Γ in $Q_{p0}Q_{0p}$!!) Thus $a-b$ determines the dimension of $[C(\Gamma, p, q) \cup \mathscr{B}(\Gamma, p, q)]$. A maximal linearly independent subset $\mathscr{A}(\Gamma, p, q)$ of $\mathscr{B}(\Gamma, p, q)$ such that $C(\Gamma, p, q) \cup \mathscr{A}(\Gamma, p, q)$ is linearly independent, is thus a basis for Γ_{pq}. This follows from theorem 2 of Gay and Ascher (1984). This basis is shown in table 10 for certain Γ.

The polynomials obtained above can be used to obtain the polynomial pairs p_{ijk} described in (4.1) and listed in table 7. In all cases $[p_{ijk}] = (CD)^{\hat{\mathbf{G}}}$ where $\mathbf{G}$ acts on C and D according to $\mathbf{E}$ and $C \subseteq P_h(\mathbf{E})$, $D \subseteq P_{rs}(\mathbf{T}_2 \oplus \mathbf{T}_2)$. Table 9 shows how to find p_{ijk} given bases for C and D. A basis for each possible C appears in table 4. A basis for each possible D is given in table 10.

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